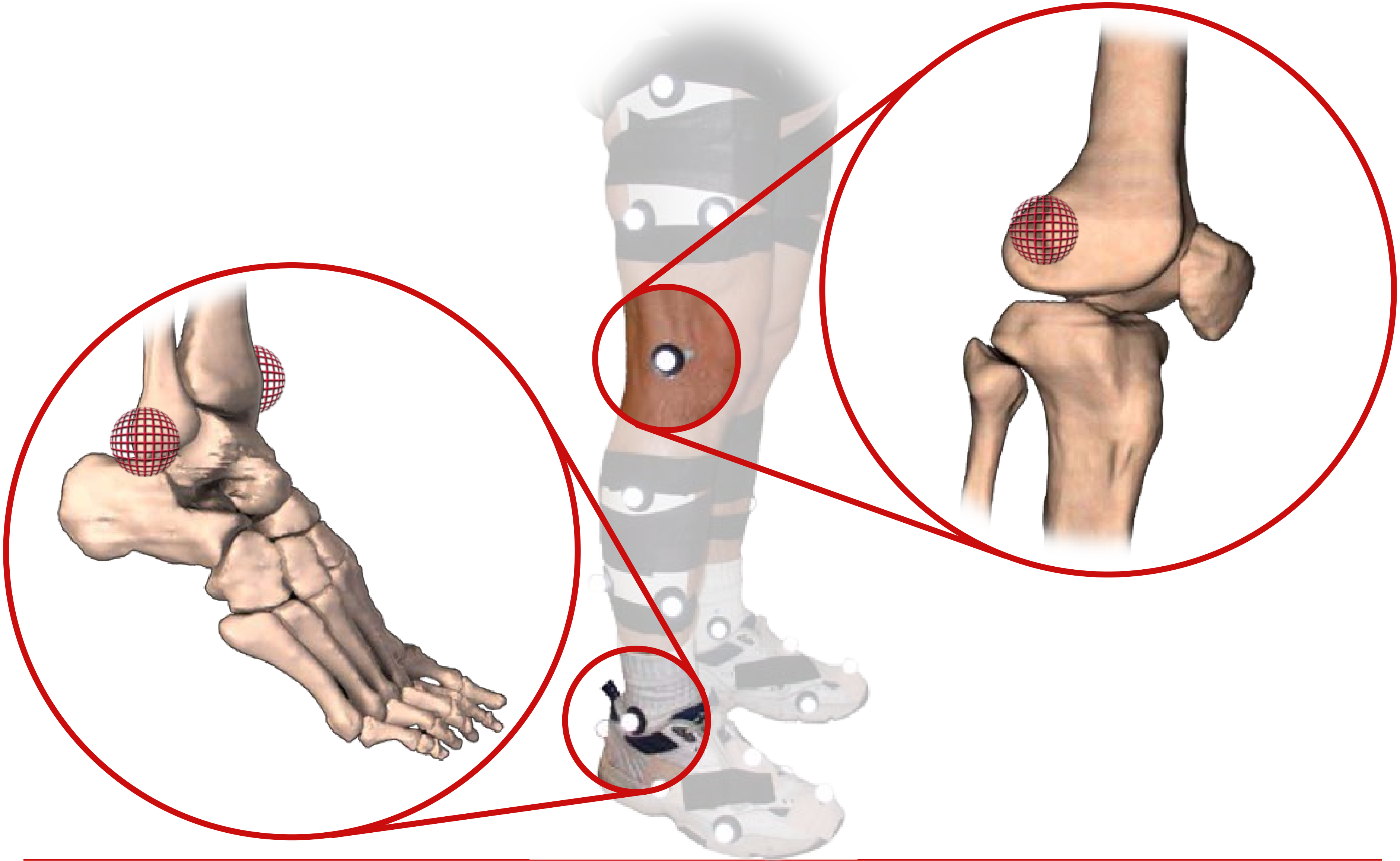
# From Markers to Joint Kinematics

1. Place markers on skeletal landmarks to define skeletal coordinate frames
2. If these markers are not convenient for motion capture, place additional markers on segments to define segment coordinate frames
3. Measure markers in lab coordinate frame
4. Determine transformation between lab & segment coordinate frames
5. Determine “static” transformation between segment & skeletal coordinate frames
6. Determine transformation between adjacent skeletal coordinate frames
7. Extract joint kinematics from transformations

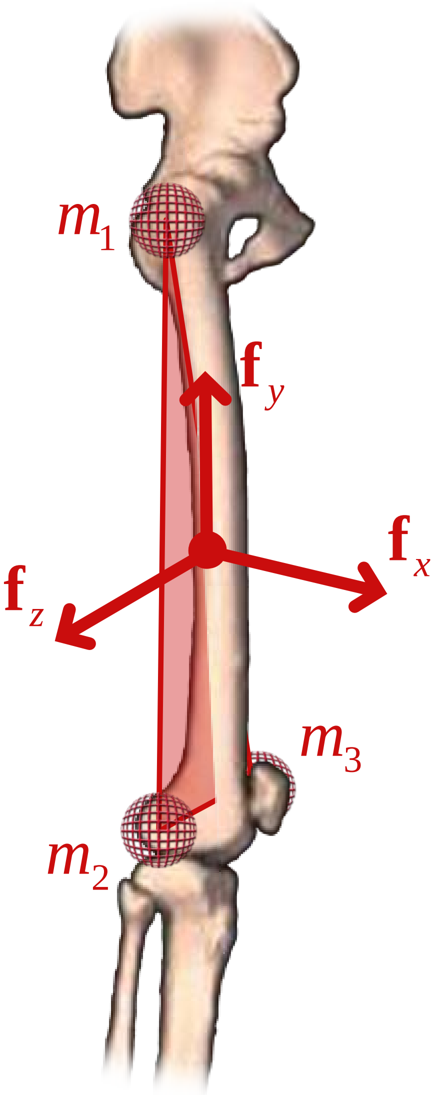
# Place Markers on Skeletal Landmarks





adapted from Martini, 2001

# Define Skeletal Coordinate Frames




$$\mathbf{v}_x = (m_3 - m_2) \times (m_1 - m_2)$$

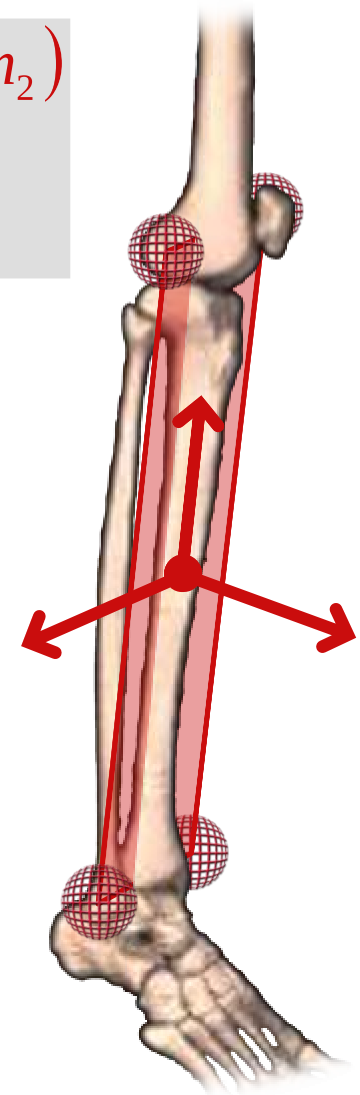
$$\mathbf{f}_x = \frac{\mathbf{v}_x}{\|\mathbf{v}_x\|}$$

$$\mathbf{v}_y = \mathbf{f}_z \times \mathbf{f}_x$$

$$\mathbf{f}_y = \frac{\mathbf{v}_y}{\|\mathbf{v}_y\|}$$

$$\mathbf{v}_z = m_2 - m_3$$

$$\mathbf{f}_z = \frac{\mathbf{v}_z}{\|\mathbf{v}_z\|}$$





adapted from Martini, 2001

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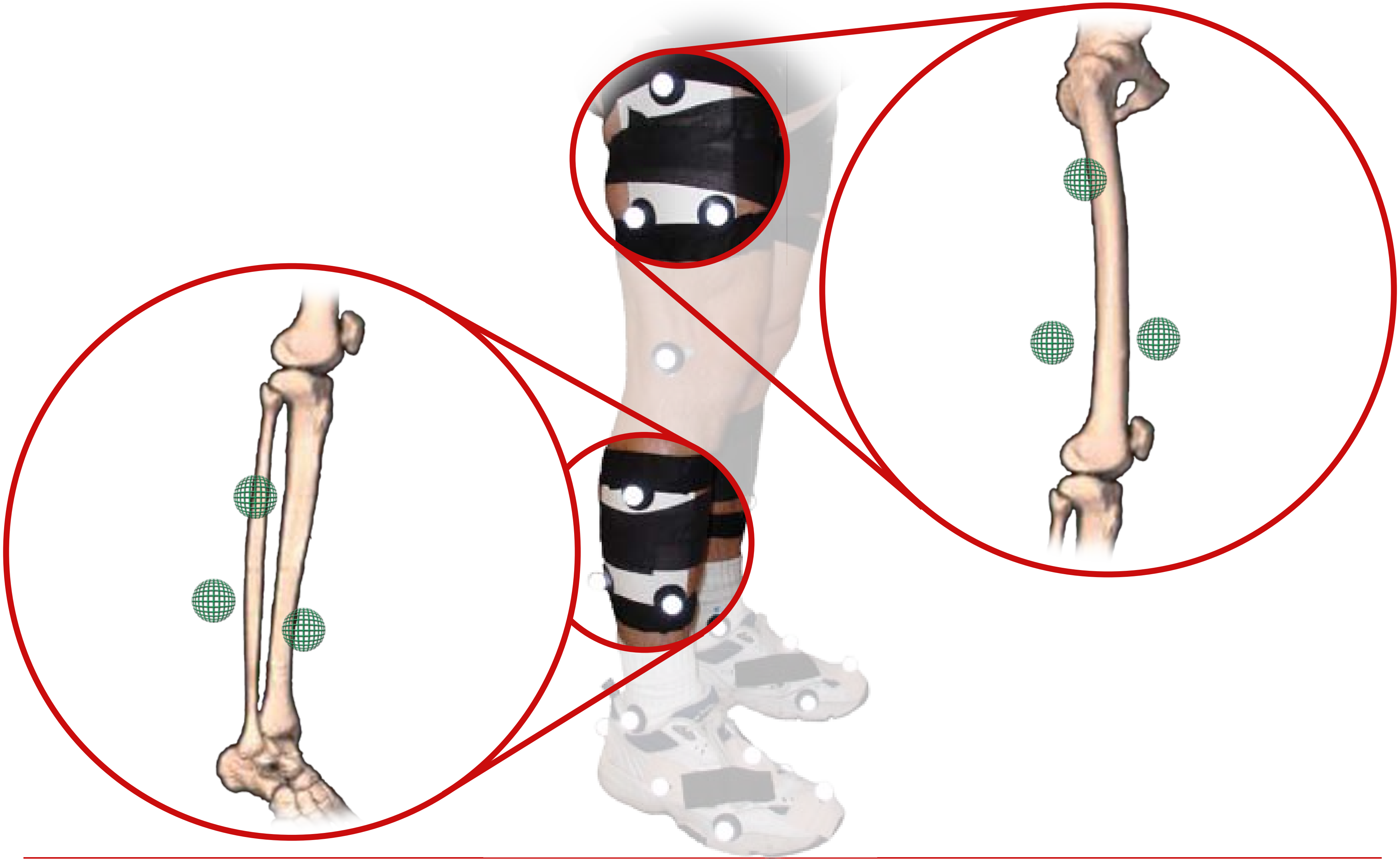
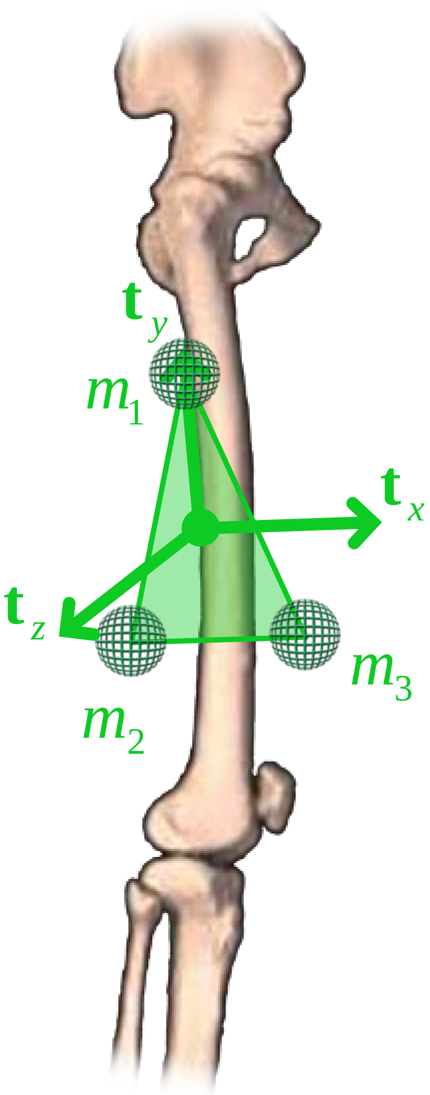
# Place Additional Markers on Segments





adapted from Martini, 2001

# Define Segment Coordinate Frames




$$\mathbf{v}_x = m_3 - m_2$$

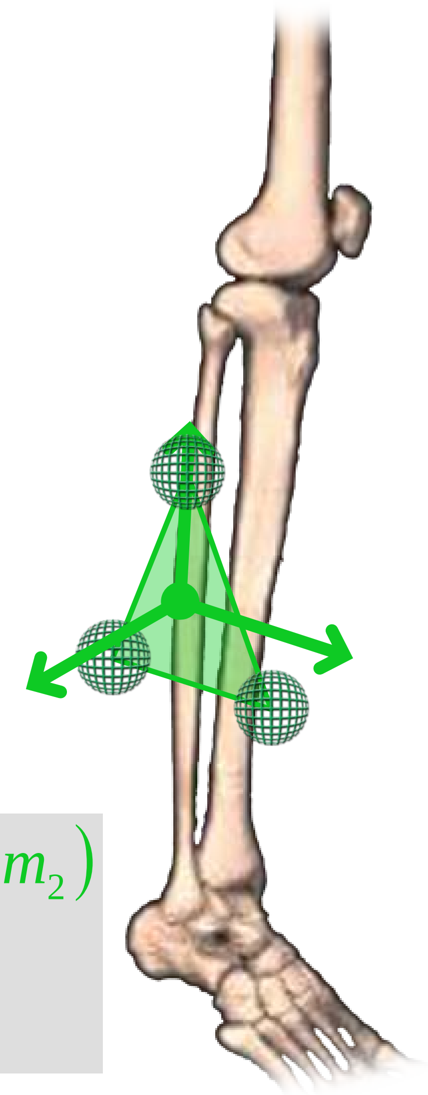
$$\mathbf{t}_x = \frac{\mathbf{v}_x}{\left\|\mathbf{v}_x\right\|}$$

$$\mathbf{v}_y = \mathbf{t}_z \times \mathbf{t}_x$$

$$\mathbf{t}_y = \frac{\mathbf{v}_y}{\left\|\mathbf{v}_y\right\|}$$

$$\mathbf{v}_z = \left(m_3 - m_2\right) \times \left(m_1 - m_2\right)$$

$$\mathbf{t}_z = \frac{\mathbf{v}_z}{\left\|\mathbf{v}_z\right\|}$$





adapted from Martini, 2001

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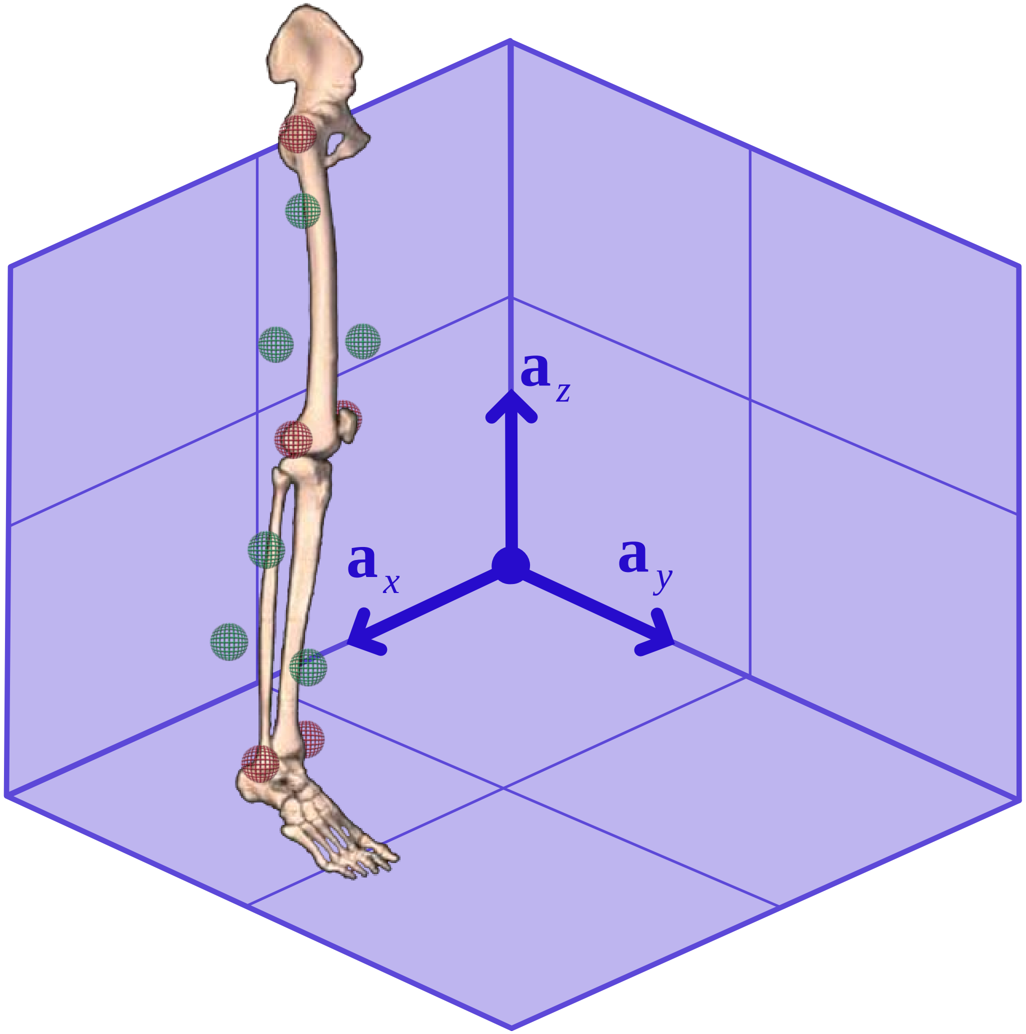
# Measure Markers in Lab Coordinate Frame






adapted from Martini, 2001

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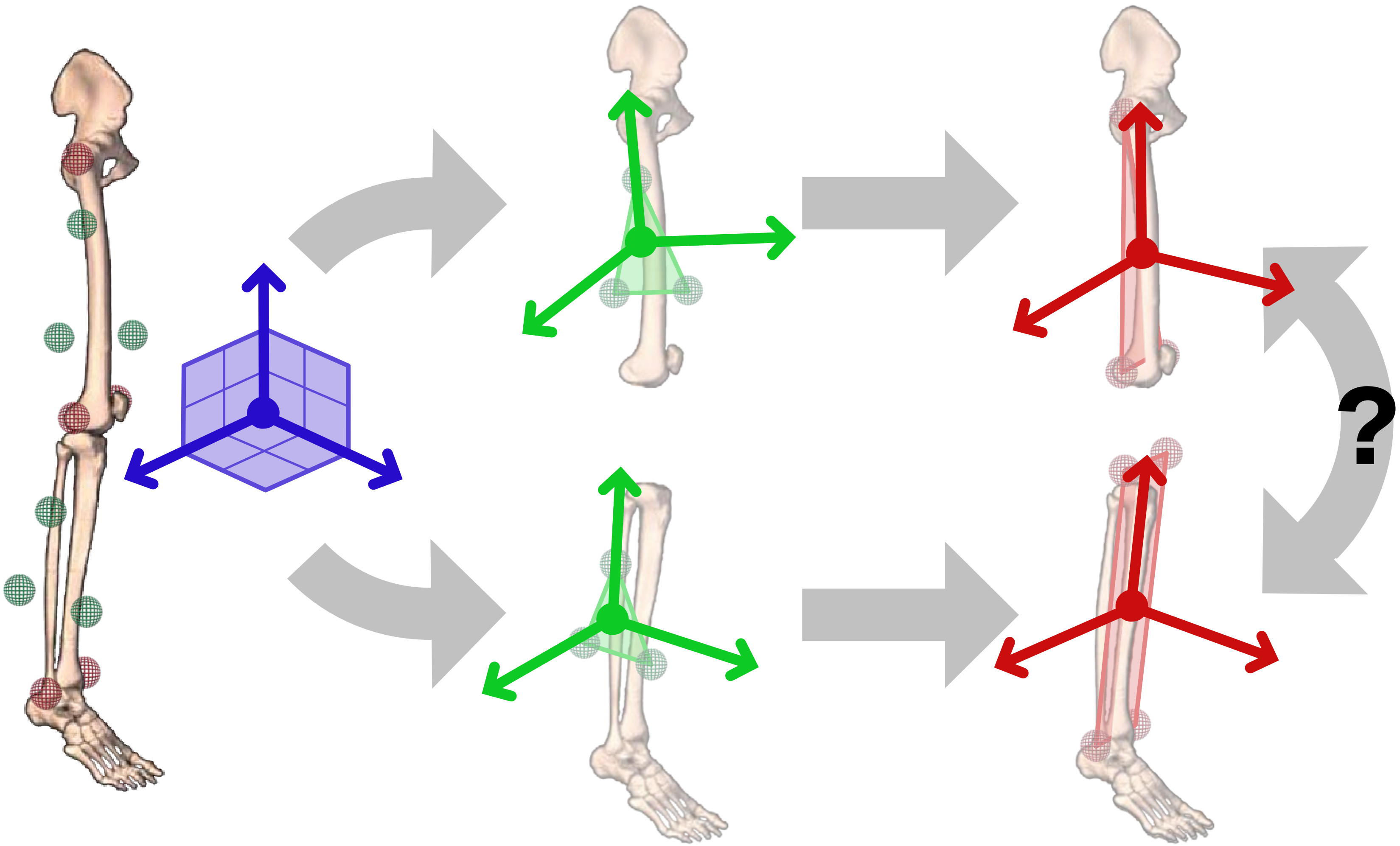
# Transformations Between Coordinate Frames





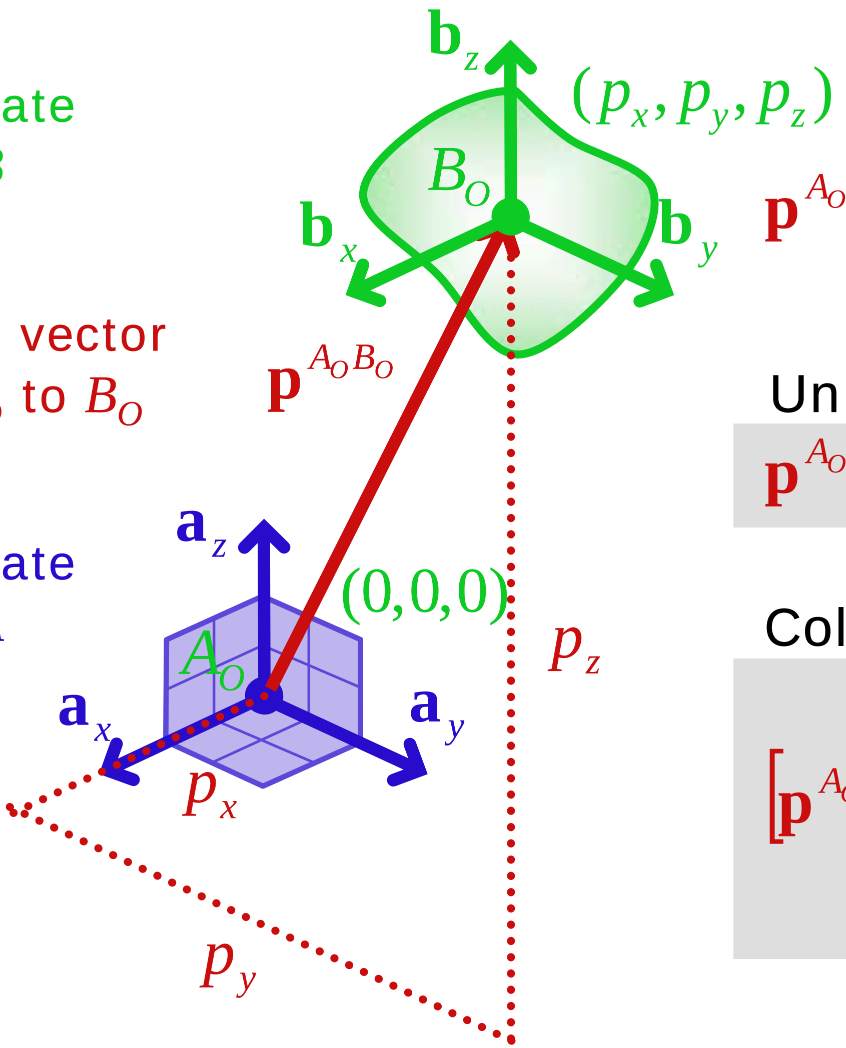
adapted from Martini, 2001

# Translation Between Coordinate Frames

Coordinate frame $B$

Position vector from $A_O$ to $B_O$

Coordinate frame $A$




$$\mathbf{p}^{A_O B_O} = B_O - A_O$$

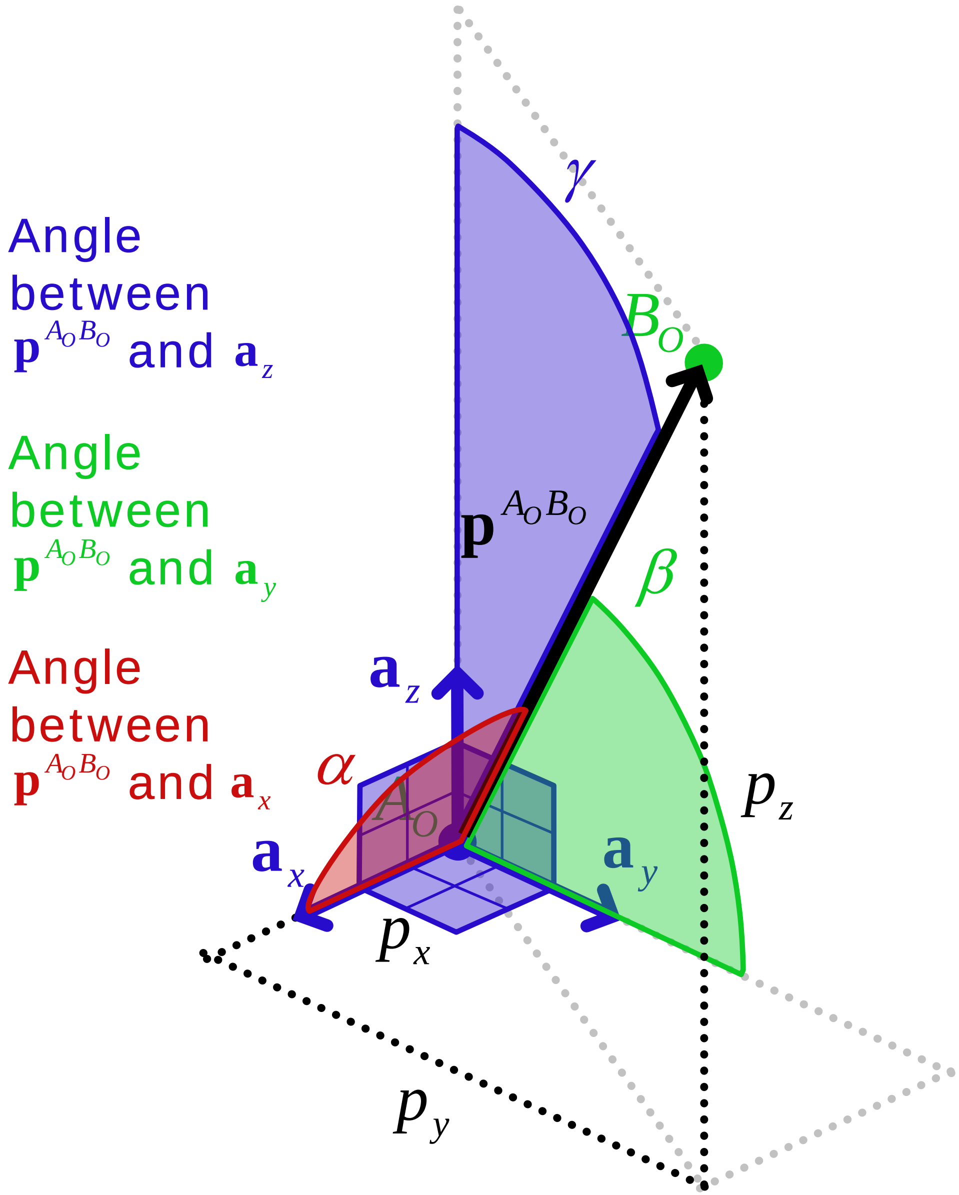
Unit vector notation

$$\mathbf{p}^{A_O B_O} = p_x \mathbf{a}_x + p_y \mathbf{a}_y + p_z \mathbf{a}_z$$

Column matrix notation

$$\left[\mathbf{p}^{A_O B_O}\right]_A = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_A$$

# Direction Cosines




Relationship between a vector and its measures

$$p_x = p \cos\alpha \ , \quad p = \left\| \mathbf{p}^{A_O B_O} \right\|$$

$$p_y = p \cos\beta$$

$$p_z = p \cos\gamma$$

Direction cosines

$$\cos\alpha = \mathbf{a}_x \cdot \mathbf{p} \ , \quad \mathbf{p} = \frac{\mathbf{p}^{A_O B_O}}{\left\| \mathbf{p}^{A_O B_O} \right\|}$$

$$\cos\beta = \mathbf{a}_y \cdot \mathbf{p}$$

$$\cos\gamma = \mathbf{a}_z \cdot \mathbf{p}$$

# Rotation Between Coordinate Frames

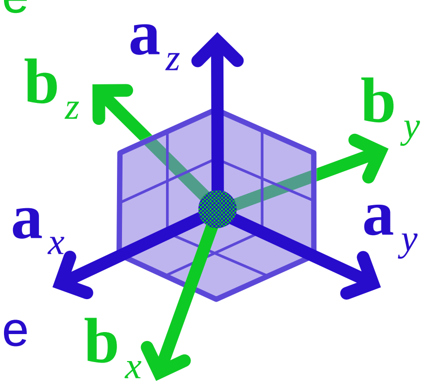
$$^{A}\mathbf{R}^{B} = \left(^{B}\mathbf{R}^{A}\right)^{-1} = \left(^{B}\mathbf{R}^{A}\right)^{T}$$

$$^{A}\mathbf{R}^{B} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{bmatrix}$$

Rotation Matrix

| $^{A}\mathbf{R}^{B}$ | $\mathbf{b}_x$ | $\mathbf{b}_y$ | $\mathbf{b}_z$ |
|---|---|---|---|
| $\mathbf{a}_x$ | $\mathbf{a}_x \cdot \mathbf{b}_x$ | $\mathbf{a}_x \cdot \mathbf{b}_y$ | $\mathbf{a}_x \cdot \mathbf{b}_z$ |
| $\mathbf{a}_y$ | $\mathbf{a}_y \cdot \mathbf{b}_x$ | $\mathbf{a}_y \cdot \mathbf{b}_y$ | $\mathbf{a}_y \cdot \mathbf{b}_z$ |
| $\mathbf{a}_z$ | $\mathbf{a}_z \cdot \mathbf{b}_x$ | $\mathbf{a}_z \cdot \mathbf{b}_y$ | $\mathbf{a}_z \cdot \mathbf{b}_z$ |

Coordinate frame *B*




Coordinate frame *A*

$$[\mathbf{v}]_A = {}^{A}\mathbf{R}^{B}[\mathbf{v}]_B$$

$$\mathbf{a}_x = (\mathbf{a}_x \cdot \mathbf{b}_x)\mathbf{b}_x + (\mathbf{a}_x \cdot \mathbf{b}_y)\mathbf{b}_y + (\mathbf{a}_x \cdot \mathbf{b}_z)\mathbf{b}_z$$

$$\mathbf{a}_y = (\mathbf{a}_y \cdot \mathbf{b}_x)\mathbf{b}_x + (\mathbf{a}_y \cdot \mathbf{b}_y)\mathbf{b}_y + (\mathbf{a}_y \cdot \mathbf{b}_z)\mathbf{b}_z$$

$$\mathbf{a}_z = (\mathbf{a}_z \cdot \mathbf{b}_x)\mathbf{b}_x + (\mathbf{a}_z \cdot \mathbf{b}_y)\mathbf{b}_y + (\mathbf{a}_z \cdot \mathbf{b}_z)\mathbf{b}_z$$

$$\mathbf{b}_x = (\mathbf{a}_x \cdot \mathbf{b}_x)\mathbf{a}_x + (\mathbf{a}_y \cdot \mathbf{b}_x)\mathbf{a}_y + (\mathbf{a}_z \cdot \mathbf{b}_x)\mathbf{a}_z$$

$$\mathbf{b}_y = (\mathbf{a}_x \cdot \mathbf{b}_y)\mathbf{a}_x + (\mathbf{a}_y \cdot \mathbf{b}_y)\mathbf{a}_y + (\mathbf{a}_z \cdot \mathbf{b}_y)\mathbf{a}_z$$

$$\mathbf{b}_z = (\mathbf{a}_x \cdot \mathbf{b}_z)\mathbf{a}_x + (\mathbf{a}_y \cdot \mathbf{b}_z)\mathbf{a}_y + (\mathbf{a}_z \cdot \mathbf{b}_z)\mathbf{a}_z$$

# Rotation Matrix for a Simple Rotation

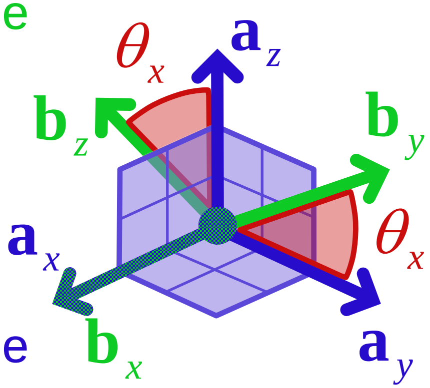
$$\mathbf{a}_x \cdot \mathbf{b}_x = \cos(0^\circ) = 1$$
$$\mathbf{a}_x \cdot \mathbf{b}_i = \cos(90^\circ) = 0,\ i = y, z$$
$$\mathbf{a}_i \cdot \mathbf{b}_x = \cos(90^\circ) = 0,\ i = y, z$$

| ${}^A\mathbf{R}^B$ | $\mathbf{b}_x$ | $\mathbf{b}_y$ | $\mathbf{b}_z$ |
|---|---|---|---|
| $\mathbf{a}_x$ | $1$ | $0$ | $0$ |
| $\mathbf{a}_y$ | $0$ | $\cos\theta_x$ | $-\sin\theta_x$ |
| $\mathbf{a}_z$ | $0$ | $\sin\theta_x$ | $\cos\theta_x$ |

Coordinate frame *B*

Rotation about $\mathbf{a}_x, \mathbf{b}_x$

Coordinate frame *A*




$$\mathbf{a}_y \cdot \mathbf{b}_y = \cos\theta_x$$
$$\mathbf{a}_y \cdot \mathbf{b}_z = \cos(90^\circ + \theta_x) = -\sin\theta_x$$
$$\mathbf{a}_z \cdot \mathbf{b}_y = \cos(90^\circ - \theta_x) = \sin\theta_x$$
$$\mathbf{a}_z \cdot \mathbf{b}_z = \cos\theta_x$$

# Rotation Matrix for a Simple Rotation (again)

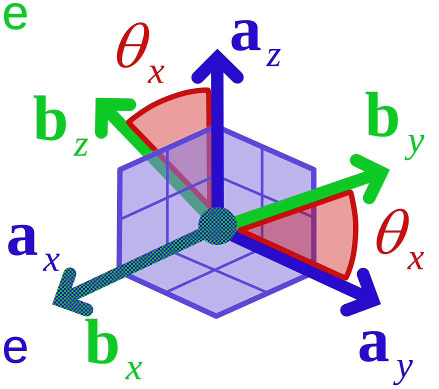
$$^{A}\mathbf{R}^{B} = \left[ [\mathbf{b}_x]_A \;\; [\mathbf{b}_y]_A \;\; [\mathbf{b}_z]_A \right]$$

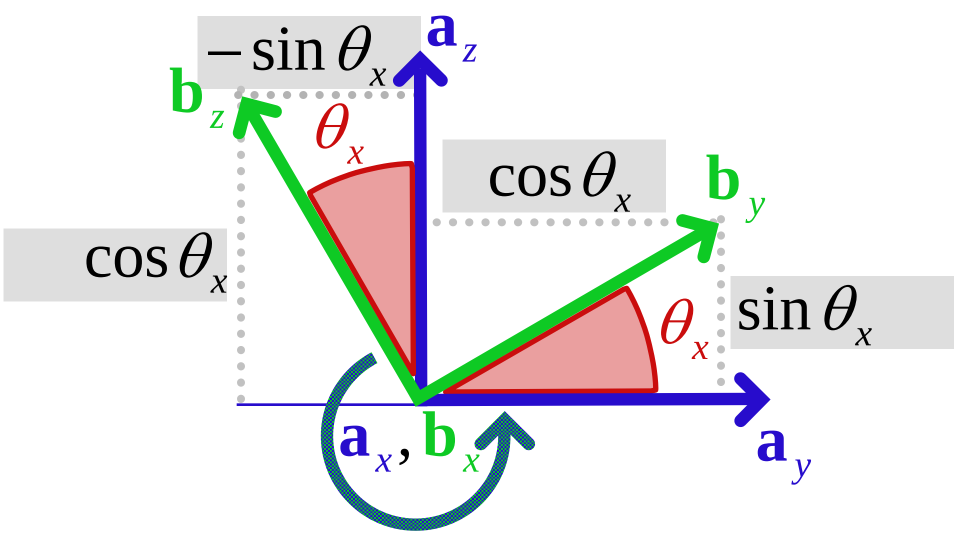
| $^{A}\mathbf{R}^{B}$ | $\mathbf{b}_x$ | $\mathbf{b}_y$ | $\mathbf{b}_z$ |
|---|---|---|---|
| $\mathbf{a}_x$ | $1$ | $0$ | $0$ |
| $\mathbf{a}_y$ | $0$ | $\cos\theta_x$ | $-\sin\theta_x$ |
| $\mathbf{a}_z$ | $0$ | $\sin\theta_x$ | $\cos\theta_x$ |

Coordinate frame $B$

Rotation about $\mathbf{a}_x, \mathbf{b}_x$

Coordinate frame $A$



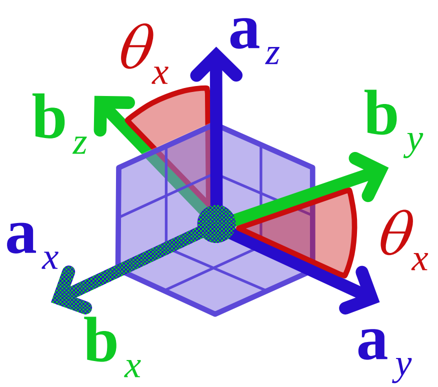
# Rotation Matrices for Simple Rotations About Each Axis

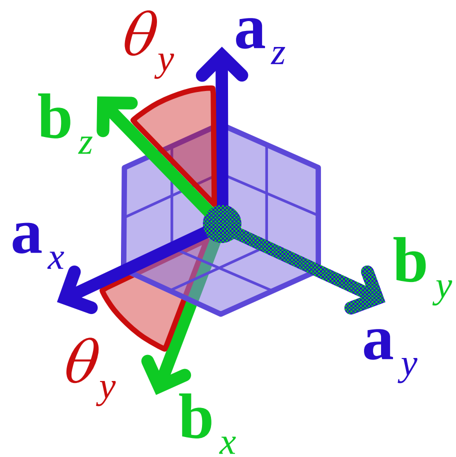
| Rotation about x-axis | Rotation about y-axis | Rotation about z-axis |
|---|---|---|

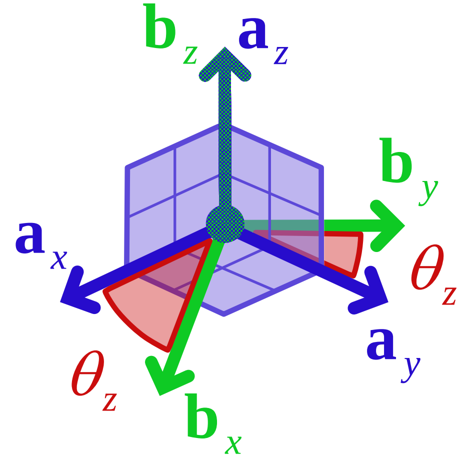
$$ {}^{A}\mathbf{R}^{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} $$

$$ {}^{A}\mathbf{R}^{B} = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} $$

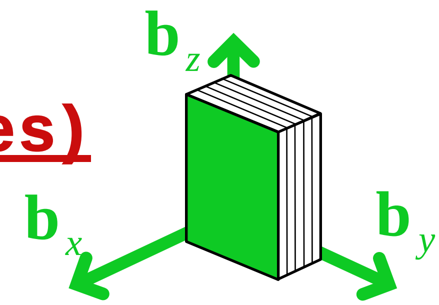
$$ {}^{A}\mathbf{R}^{B} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

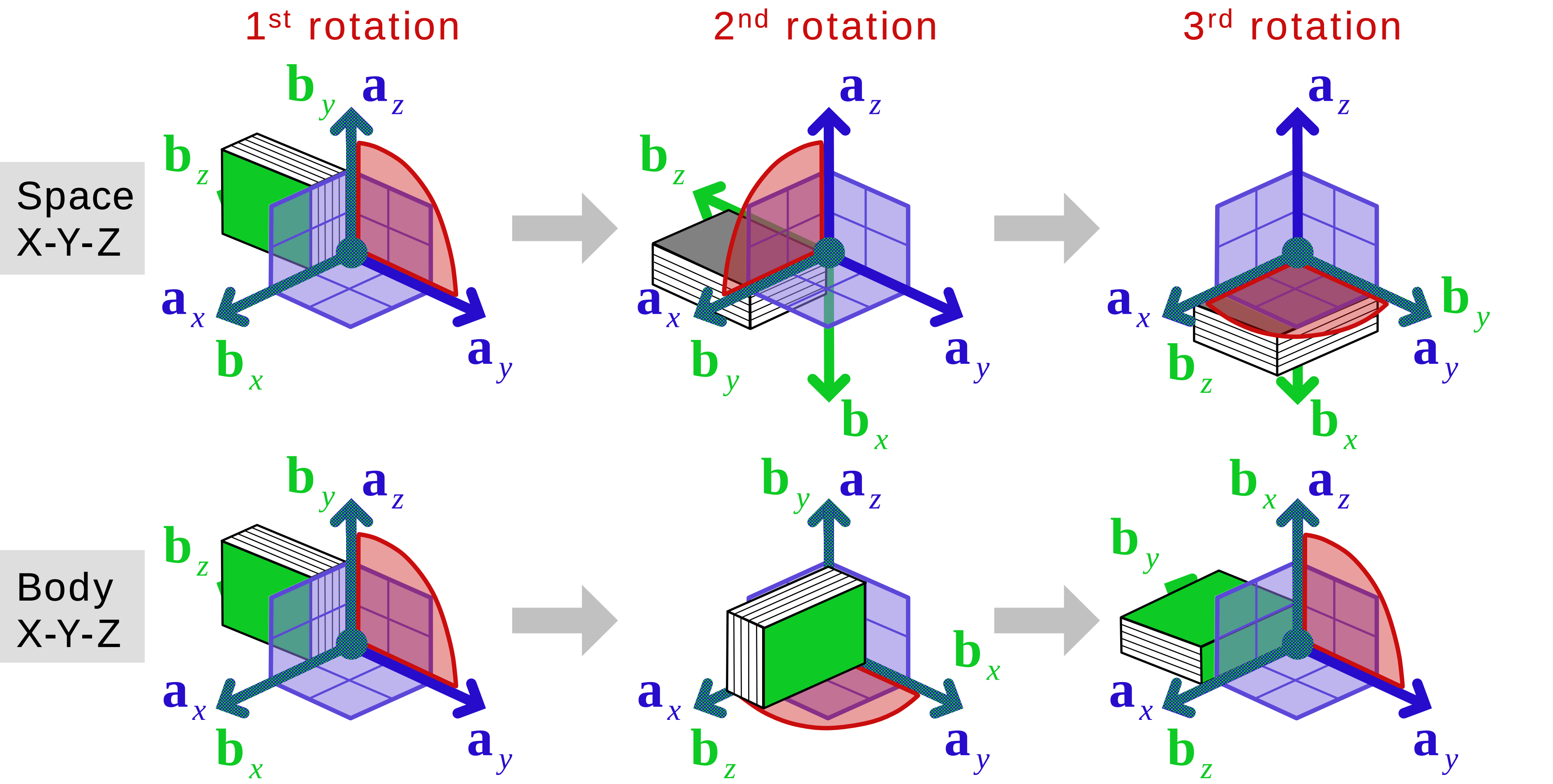


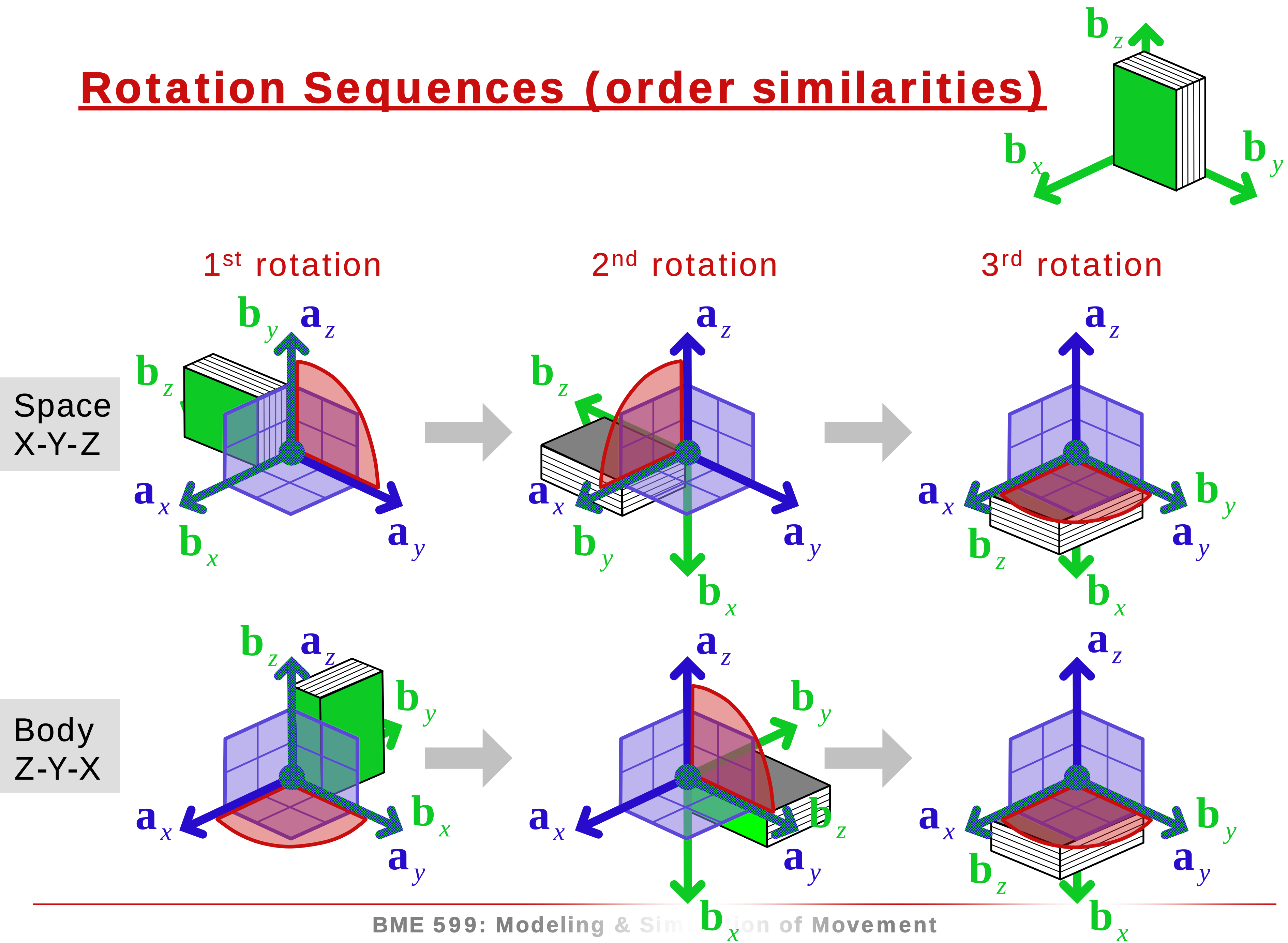



# Rotation Sequences (order differences)

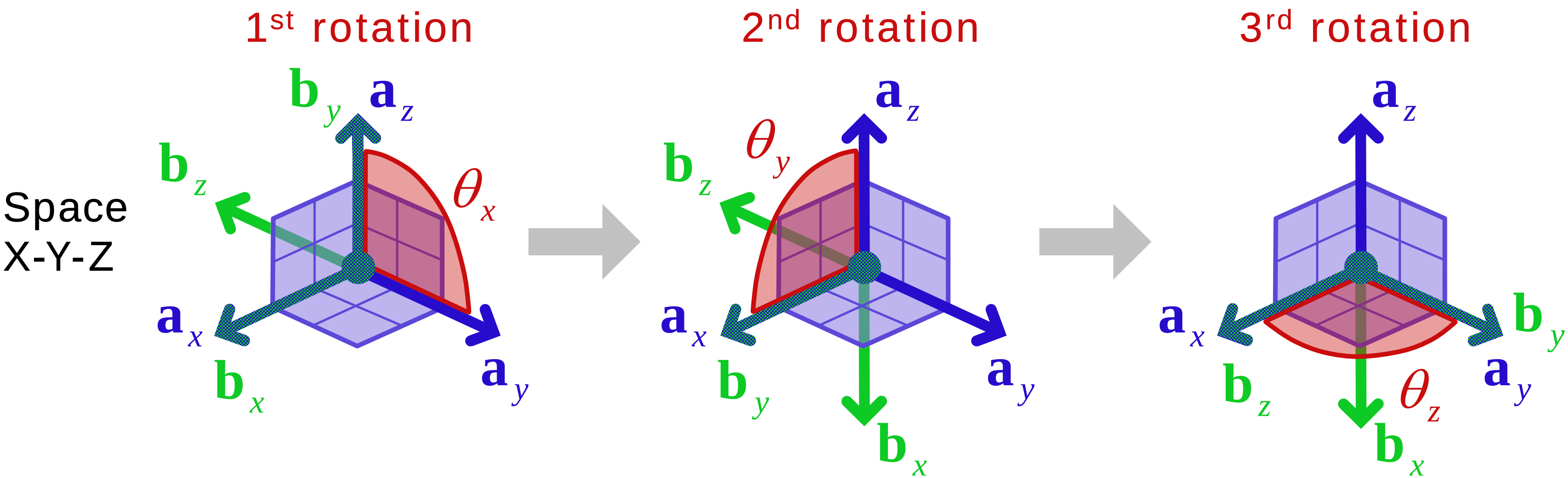


# Rotation Sequences (order similarities)



# Rotation Matrix for Space X-Y-Z Rotation Sequence




$$
{}^{A}\mathbf{R}^{B} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}
$$

$$
{}^{A}\mathbf{R}^{B} = \begin{bmatrix} c_y c_z & s_x s_y c_z - s_z c_x & s_x s_z + s_y c_x c_z \\ s_z c_y & c_x c_z + s_x s_y s_z & s_y s_z c_x - s_x c_z \\ -s_y & s_x c_y & c_x c_y \end{bmatrix}
$$

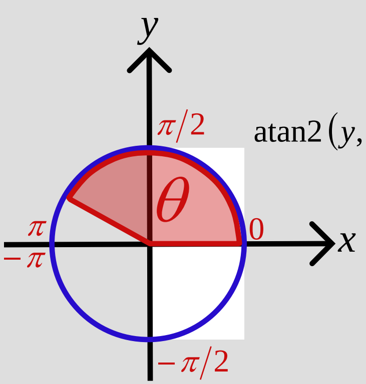
# Rotation Angles From a Rotation Matrix

$$ {}^{A}\mathbf{R}^{B} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{bmatrix} \qquad \left({}^{A}\mathbf{R}^{B}\right)_{SpaceXYZ} = \begin{bmatrix} c_y c_z & s_x s_y c_z - s_z c_x & s_x s_z + s_y c_x c_z \\ s_z c_y & c_x c_z + s_x s_y s_z & s_y s_z c_x - s_x c_z \\ -s_y & s_x c_y & c_x c_y \end{bmatrix} $$

Direction cosines are independent of rotation sequence

Rotation angles are dependent on rotation sequence

Four-quadrant inverse tangent




$$ \operatorname{atan2}(y,x) = \begin{cases} \operatorname{atan}\left(\frac{y}{x}\right) & x > 0 \\ \pi + \operatorname{atan}\left(\frac{y}{x}\right) & y \ge 0, x < 0 \\ -\pi + \operatorname{atan}\left(\frac{y}{x}\right) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases} $$

$$ \theta_x = \operatorname{atan2}\left(\frac{r_{zy}}{c_y}, \frac{r_{zz}}{c_y}\right), \quad \begin{cases} r_{zy} = s_x c_y \\ r_{zz} = c_x c_y \end{cases} $$

$$ \theta_y = \operatorname{atan2}\left(-r_{zx}, \sqrt{r_{xx}^2 + r_{yx}^2}\right), \quad \begin{cases} r_{zx} = -s_y \\ r_{xx} = c_y c_z \\ r_{yx} = s_z c_y \end{cases} $$

$$ \theta_z = \operatorname{atan2}\left(r_{yx}, r_{xx}\right) $$

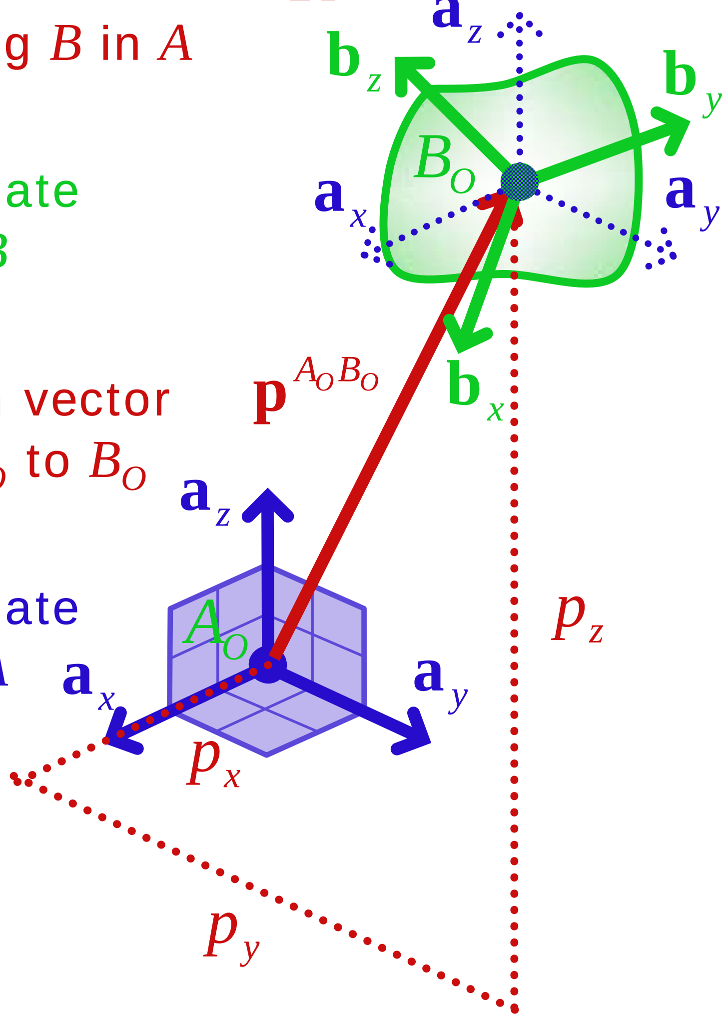
# Transformation Between Coordinate Frames

Rotation matrix orienting $B$ in $A$ — ${}^{A}\mathbf{R}^{B}$

Coordinate frame $B$

Position vector from $A_O$ to $B_O$ — $\mathbf{p}^{A_O B_O}$

Coordinate frame $A$

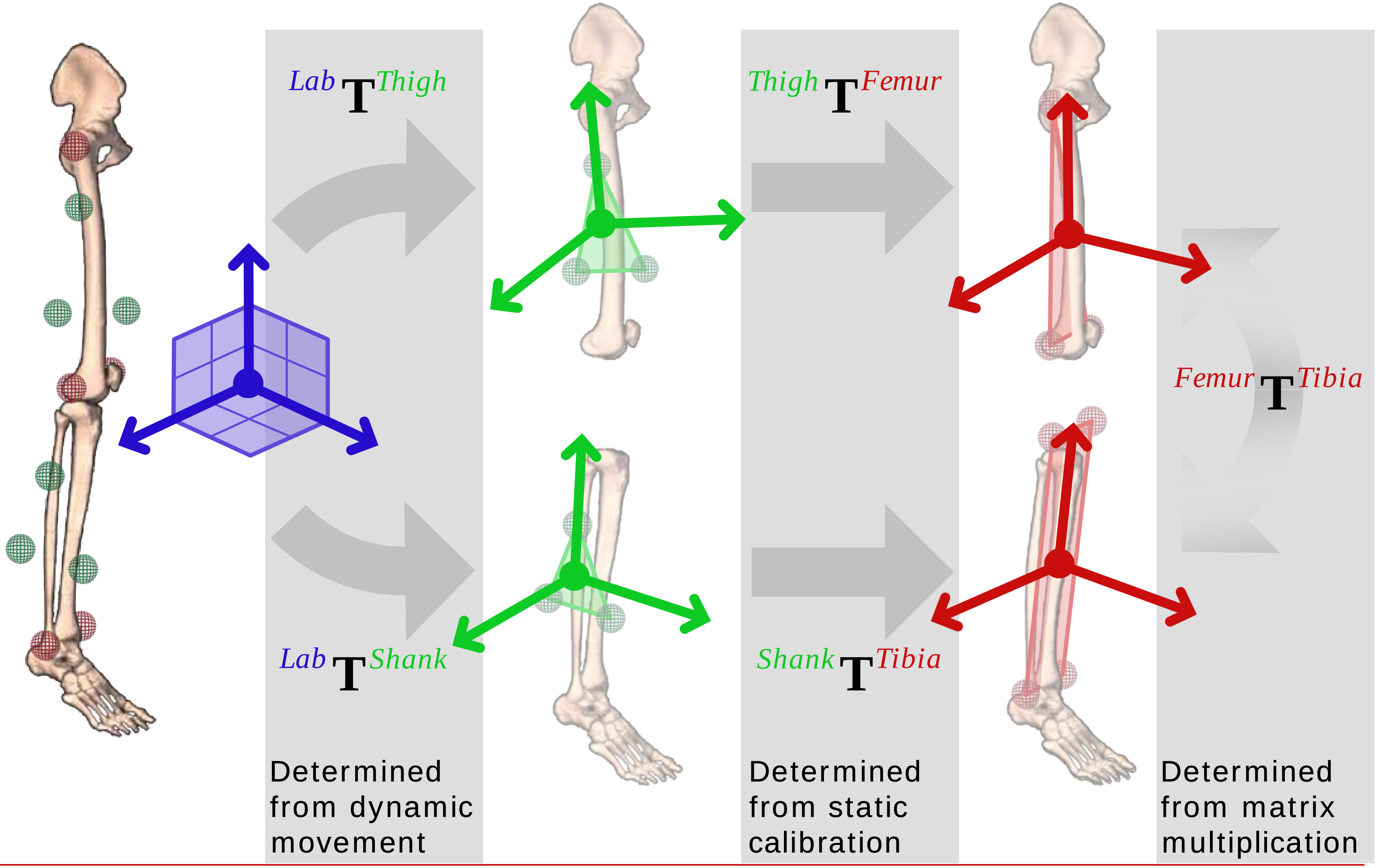



Transformation matrix

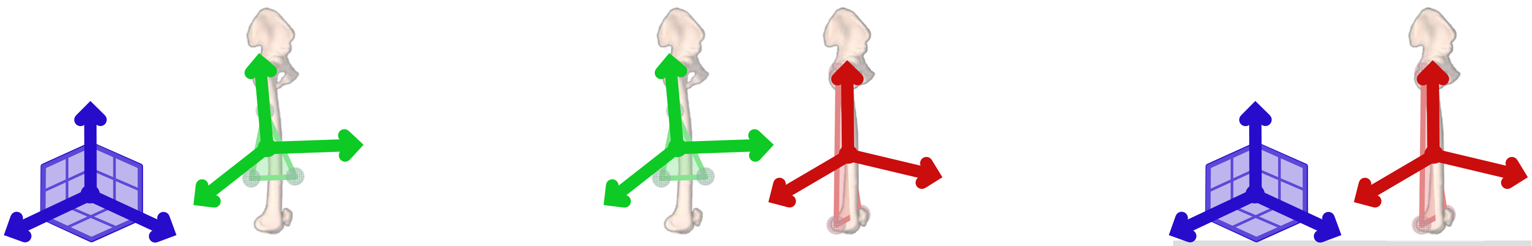
$$
{}^{A}\mathbf{T}^{B} = \begin{bmatrix} {}^{A}\mathbf{R}^{B} & \left[\mathbf{p}^{A_O B_O}\right]_A \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}
$$

$$
{}^{A}\mathbf{T}^{B} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} & p_x \\ r_{yx} & r_{yy} & r_{yz} & p_y \\ r_{zx} & r_{zy} & r_{zz} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

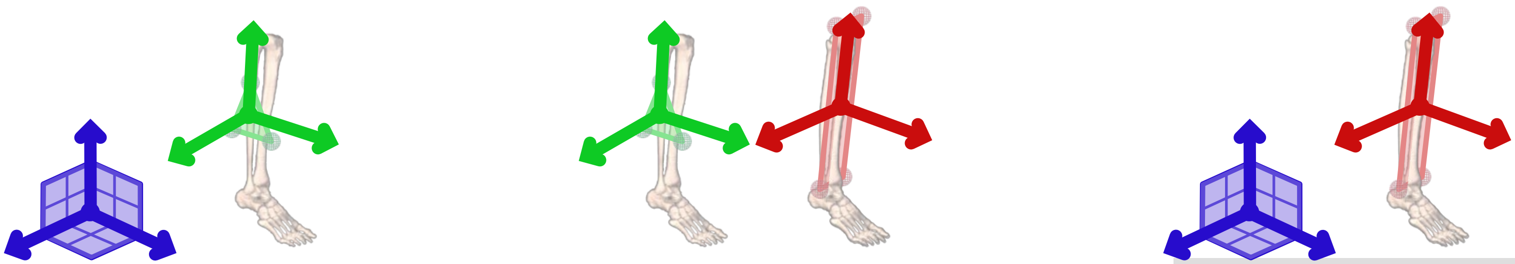
# Determining Kinematics from Transformations



# Determining Kinematics from Transformations

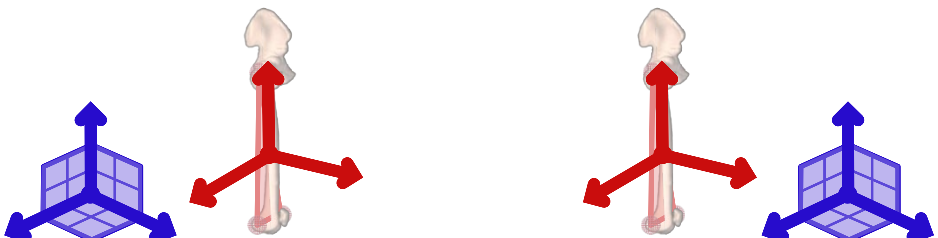


$${}^{Lab}\mathbf{T}^{Thigh} * {}^{Thigh}\mathbf{T}^{Femur} = {}^{Lab}\mathbf{T}^{Femur}$$

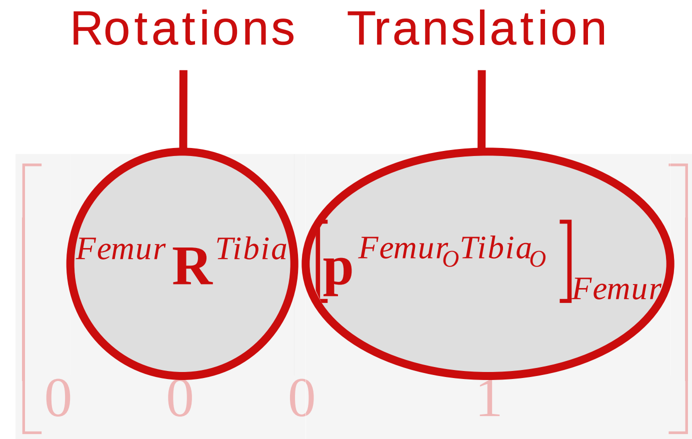


$${}^{Lab}\mathbf{T}^{Shank} * {}^{Shank}\mathbf{T}^{Tibia} = {}^{Lab}\mathbf{T}^{Tibia}$$

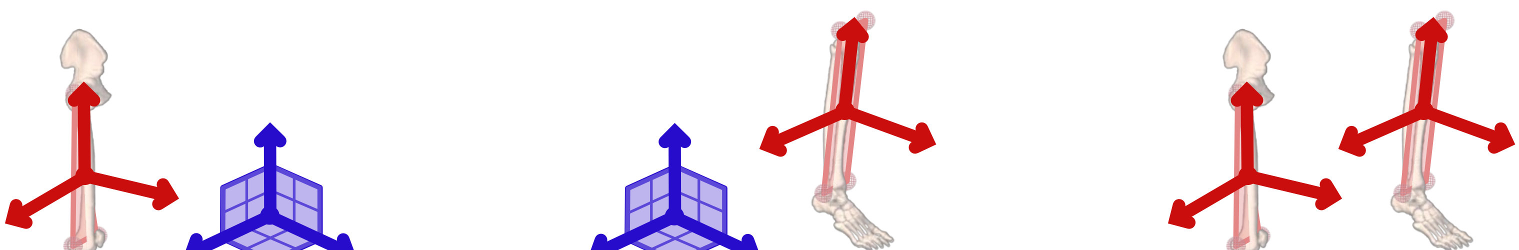
# Determining Kinematics from Transformations



$$\left({}^{Lab}\mathbf{T}^{Femur}\right)^{-1} = {}^{Femur}\mathbf{T}^{Lab}$$




$$\begin{bmatrix} {}^{Femur}\mathbf{R}^{Tibia} & \left[\mathbf{p}^{Femur_O Tibia_O}\right]_{Femur} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$



$${}^{Femur}\mathbf{T}^{Lab} * {}^{Lab}\mathbf{T}^{Tibia} = {}^{Femur}\mathbf{T}^{Tibia}$$