DYNAMICS SUMMARY

Two different free-body methods for deriving dynamics equations for systems of rigid bodies connected by joints are described briefly below – the Newton-Euler method and the Motion Law (sometimes referred to as D'Alembert's method). The goal is to provide a big picture understanding of these methods without getting lost in the details.

I. NEWTON-EULER METHOD

The Newton-Euler method of deriving dynamics equations involving treating every body as a separate "free body" and solving for ALL reaction forces and torques. Thus, for a 2D problem, you will need to form 3 equations (2 translations and 1 rotational) for every body, while for a 3D problem, you will need to form 6 equations (3 translational and 3 rotational) for each body. If *nbodies* is the number of bodies, this means that the total number of unknowns will always be 3**nbodies* for a 2D problem and 6**nbodies* for a 3D problem. These unknowns will be combinations of $\ddot{q}$'s and reaction forces and torques. For example, a 2D problem with a single rigid segment connected to ground by a pin joint will have 3 unknowns that must be solved – one $\ddot{q}$ and two reaction forces (horizontal and vertical). The translational equations come directly from the linear momentum principle, while the rotational equations come directly from the angular momentum principle where the mass center must be chosen as the point about which to sum moments.

II. THE MOTION LAW

The Motion Law (often attributed incorrectly to D'Alembert) replaces the linear and angular momentum principles for a free body S (S can be a single body or group of bodies) with the related equations

$$\mathbf{F}^{S}_{Contact} + \mathbf{F}^{S}_{Distance} + \mathbf{F}^{S}_{Inertia} = \mathbf{0}$$
$$\mathbf{M}^{S/P}_{Contact} + \mathbf{M}^{S/P}_{Distance} + \mathbf{M}^{S/P}_{Inertia} = \mathbf{0}$$

where $\mathbf{F}^{S}_{Contact}$ are all contact forces acting on S (e.g., ground contact forces), $\mathbf{F}^{S}_{Distance}$ are all distance forces acting on S (e.g., gravity forces), and $\mathbf{F}^{S}_{Inertia}$ are all inertia forces acting on S. For any individual body B, $\mathbf{F}^{B}_{Inertia}$ can be easily calculate as

$$\mathbf{F}^{B}_{Inertia} = -m_B \, {}^{N}\mathbf{a}^{B}$$

which is equivalent to $-{}^{N}d\,{}^{N}\mathbf{L}^{B}/dt$ for body B.

As indicated above, these contact, distance, and inertia forces also have moment counterparts. However, unlike the angular momentum principle, the moments may be taken about ANY point P. This is the primary advantage of the Motion Law over the Newton-Euler method. This advantage eliminates the need to solve large systems of equations for large numbers of unwanted reaction forces and torques. Furthermore, it makes the formulation and solution of dynamics problems identical conceptually to the solution of statics problems. The contact moment $\mathbf{M}_{Contact}^{S/P}$ is simply the moment about point *P* of all contact forces acting on free body *S*. Similarly, the distance moment $\mathbf{M}_{Distance}^{S/P}$ is the moment about point *P* of all distance forces acting on *S*. The inertia moment $\mathbf{M}_{Inertia}^{S/P}$ is the quantity that is different, as it is composed of two contributions: the moment of all inertia forces about point *P*, plus a new quantity called an inertia torque. For a rigid body *B*, the inertia torque is defined as

$$\mathbf{T}^{B/B*} = -{}^{N}\boldsymbol{\alpha}^{B} \cdot \mathbf{I}^{B/B*} - {}^{N}\boldsymbol{\omega}^{B} \times \mathbf{I}^{B/B*} \cdot {}^{N}\boldsymbol{\omega}^{B}$$

which is equivalent to $-{}^{N}d\,{}^{N}\mathbf{H}^{B/B*}/dt$ for body *B* based on the Million Dollar Formula. Thus, the inertia moment $\mathbf{M}_{Inertia}^{B/P}$ for body *B* about point *P* becomes

$$\mathbf{M}_{Inertia}^{B/P} = \mathbf{p}^{PB*} \times \mathbf{F}_{Inertia}^{B} + \mathbf{T}^{B/B*}$$

Note that if point *P* is taken as $B*$, the mass center of body *B*, then the cross product above goes to zero, and the inertia moment becomes identical to the inertia torque.

For a more thorough discussion of the Moton Law, consult Kane and Levinson (1997 and 1998).

III. SOLUTION PROCESS

1) Define reference frames, unit vectors fixed in each reference frame, and unknown translations and rotations (i.e., generalized coordinates) corresponding to the degrees of freedom.
2) Draw free body diagrams, realizing that groups of individual bodies can be treated as a "free body."
3) Apply the Newton-Euler, Motion Law, or Kane method, choosing directions in which to sum forces and points about which to sum moments in such as way as to minimize the need to calculate unwanted reaction forces and moments. Create a table listing which body or group of bodies will be treated as the "free body," which equation is to be used (e.g., $\mathbf{F}$ to represent sum of forces and $\mathbf{M}^{P}$ to represent the moment about some point *P*), in which direction a dot product will be taken, and finally what the resulting unknowns will be (i.e., second time derivatives of the unknown translations and rotations and possibly reaction forces and torques).
4) Calculate important kinematic quantities (i.e., the acceleration of every point with mass, the angular velocity and acceleration of every body with inertia) with respect to the inertial reference frame and express them in terms of time derivatives of the unknown translations and rotations.
5) Solve the resulting linear system of equations for the second time derivatives of the unknown translations and rotations and for the unknown reaction forces and torques (if present).
6) Integrate the second time derivative equations numerically to determine the unknown translations and rotations as a function of time.