General Plane Motion

Lecture 26

ME 231: Dynamics

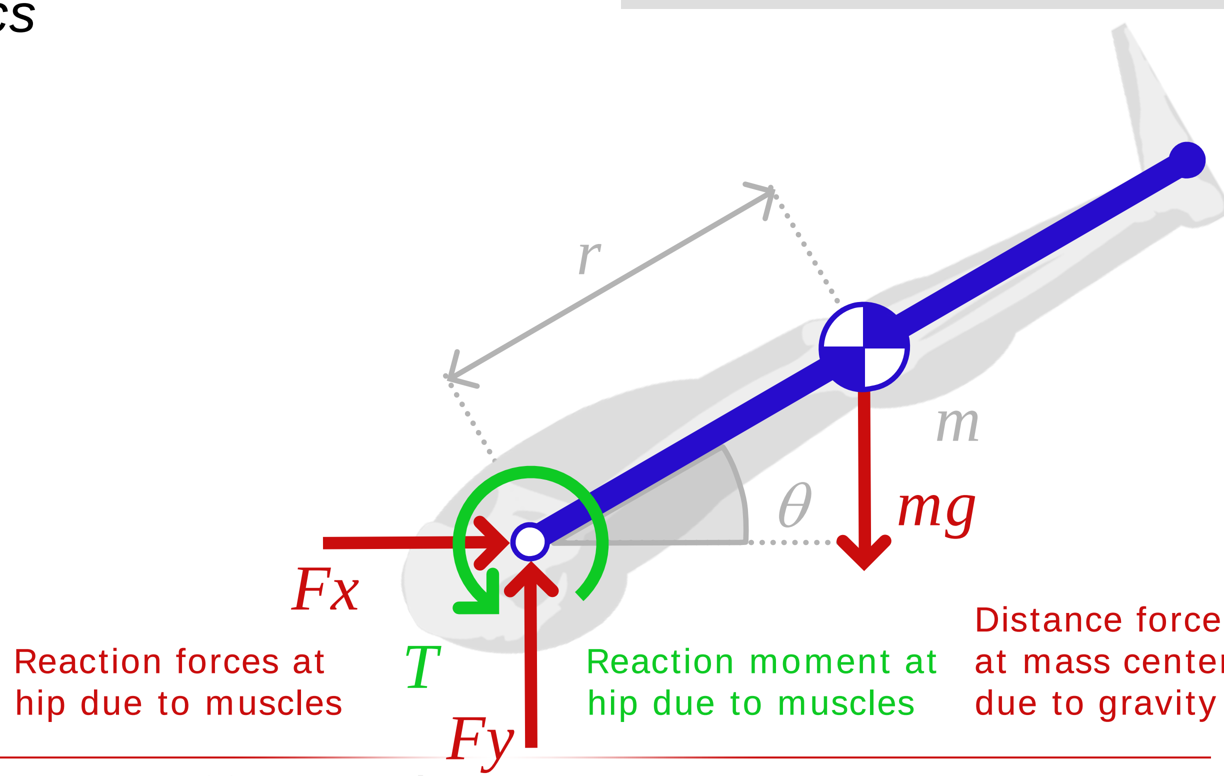
What Additional Steps Are Required for Inverse Dynamics?

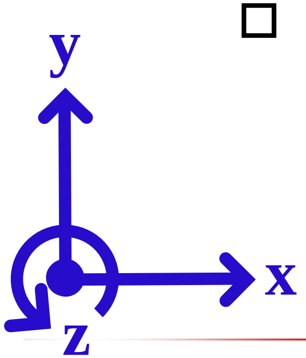
1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐
2. Hint: *kinematics*
   - ☐
   - ☐
   - ☐
3. Hint: *kinetics*
   - ☐
   - ☐

Kinematics: study of motion of a body without considering forces that cause that motion

Kinetics: study of how external forces contribute to the motion of a body

Dynamics = kinematics + kinetics





What Additional Steps Are Required for Inverse Dynamics?

1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐ Inertia forces & moments
2. Form motion quantities (kinematics)
   - ☐ Positions $(x,\ y,\ \theta)$
   - ☐ Velocities $(\dot{x},\ \dot{y},\ \dot{\theta})$
   - ☐ Accelerations $(\ddot{x},\ \ddot{y},\ \ddot{\theta})$
3. Apply Newton's 2nd Law (kinetics)
   - ☐ Forces $(\Sigma F_x = m\ddot{x})$, $(\Sigma F_y = m\ddot{y})$
   - ☐ Moments $(\Sigma M = I\ddot{\theta})$

<u>Kinematics</u>: study of motion of a body without considering forces that cause that motion

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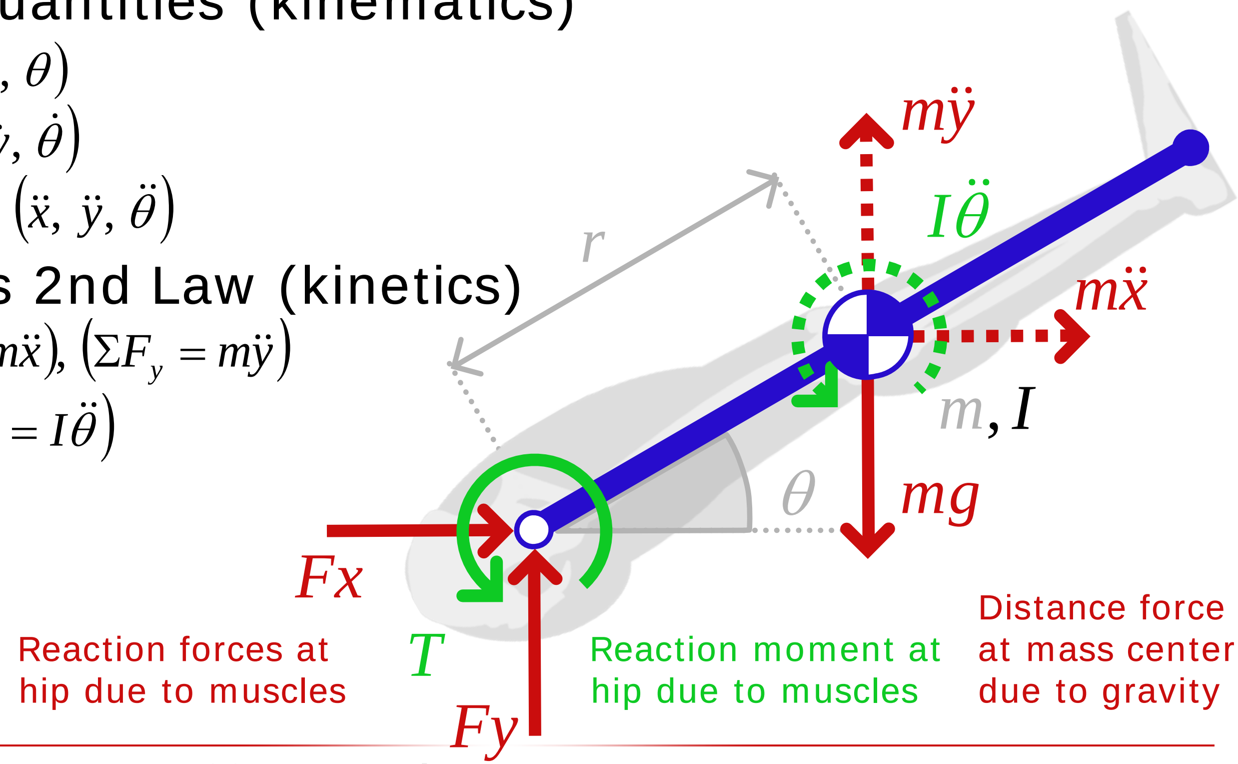
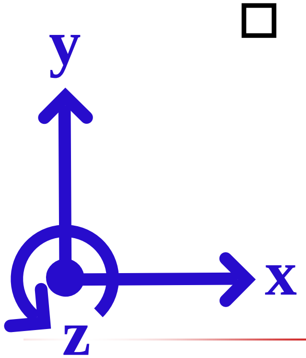
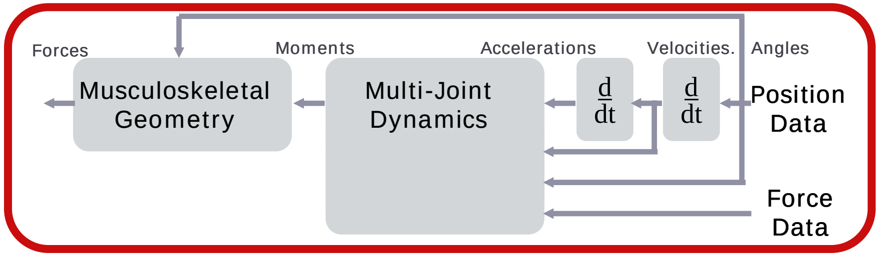
Dynamics = kinematics + kinetics

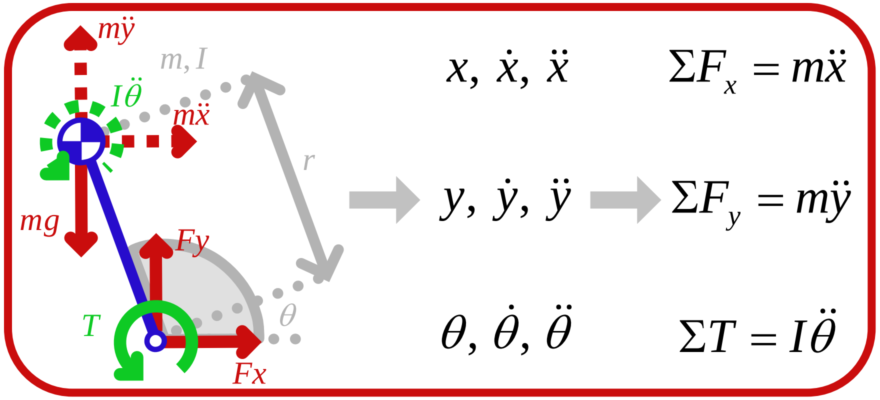




Plan for Today

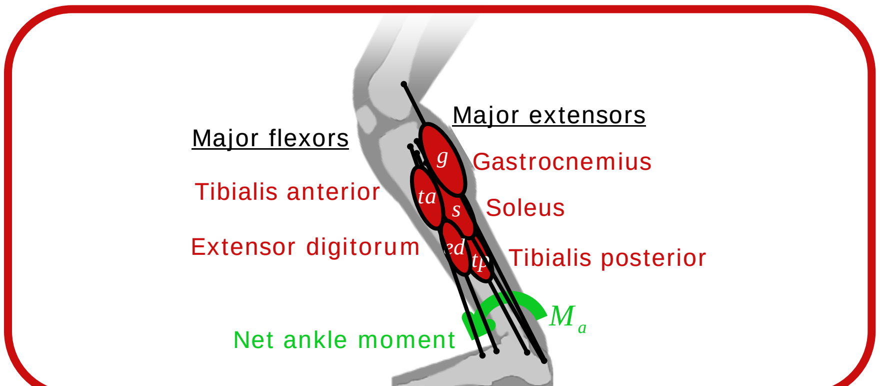


Quick review of the inverse dynamics problem



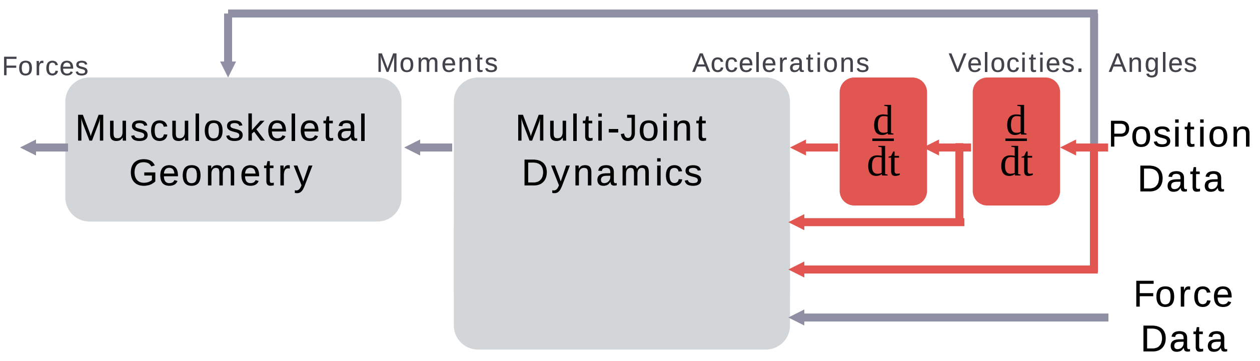
3 steps for inverse dynamics

- Free body diagram
- Kinematics
- Kinetics



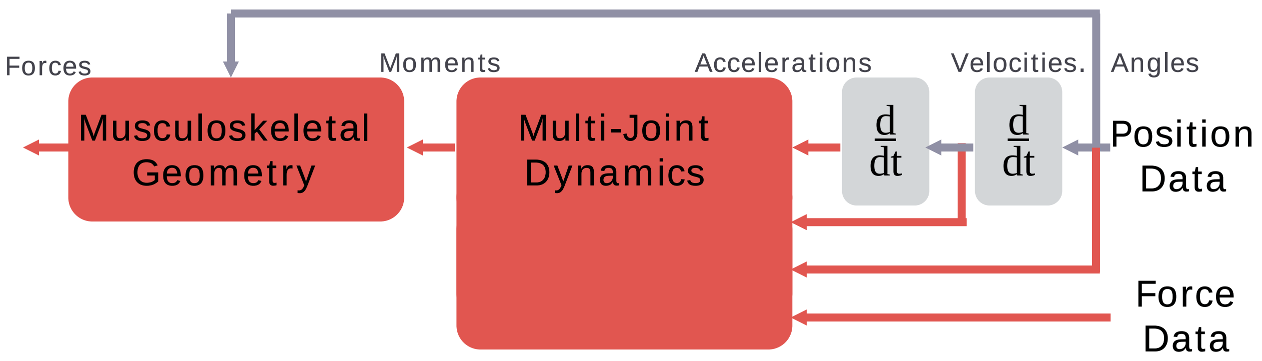
Distribution problem

The Inverse Dynamics Problem



- ✓ Identify question that you will use inverse dynamics to answer
- ✓ Based on question, determine DOFs to be measured and modeled
- ✓ Measure joint kinematics
- ✓ Filter and differentiate kinematics data

The Inverse Dynamics Problem



- Derive equations of motion from model of system
- Solve equations of motion with and without external forces
- Use musculoskeletal geometry and assumptions about force distribution to estimate individual muscle forces

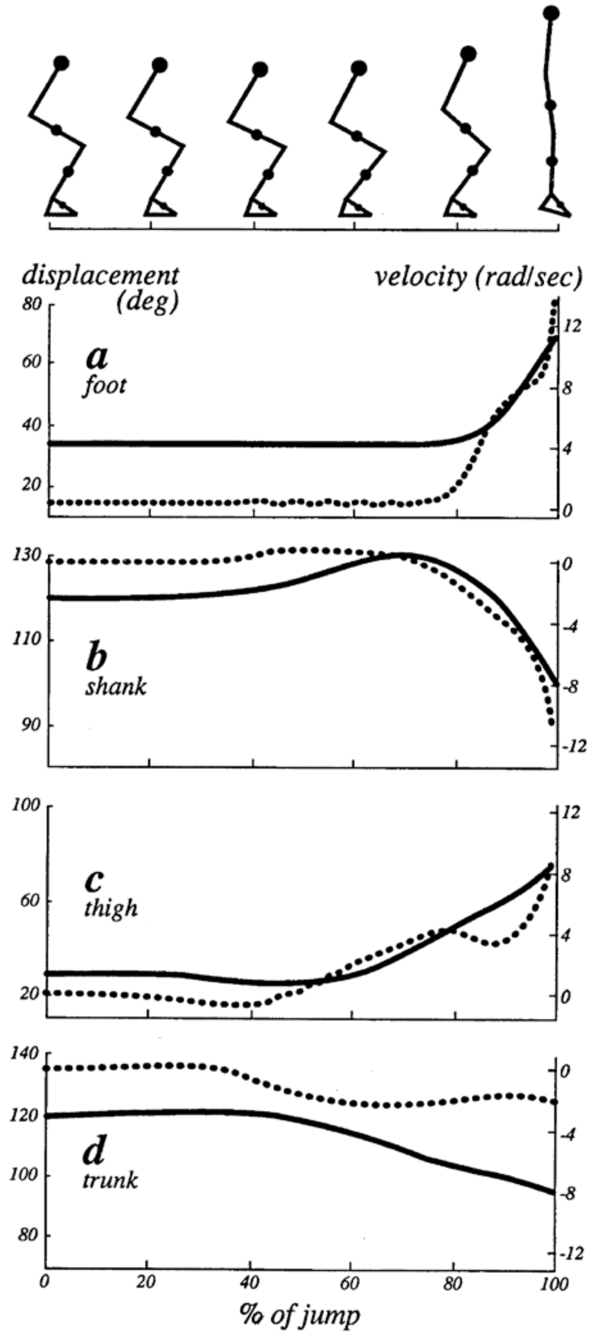
A Possible Question

What are the sagittal plane moments about the ankle, knee, and hip during a maximum height jump?

Experimental set-up

The Experimental Results

Experiments provide joint angles, angular velocities, and ground reaction forces during movement

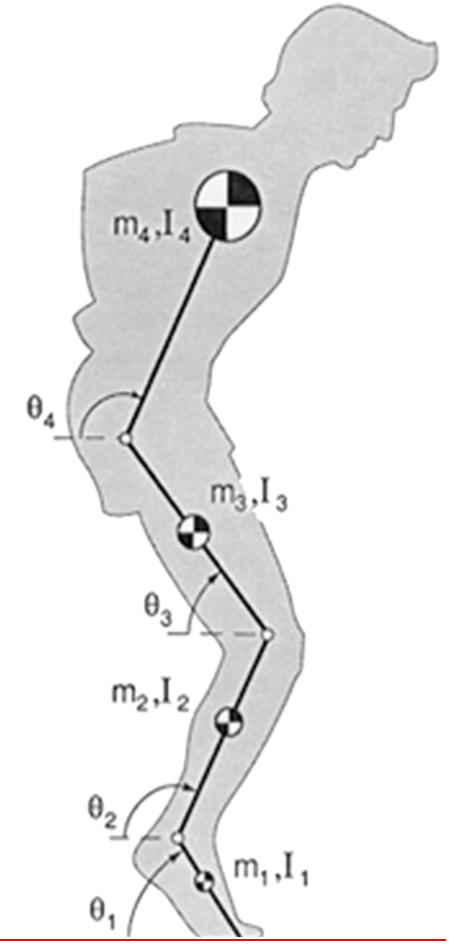


Model of System

Must include

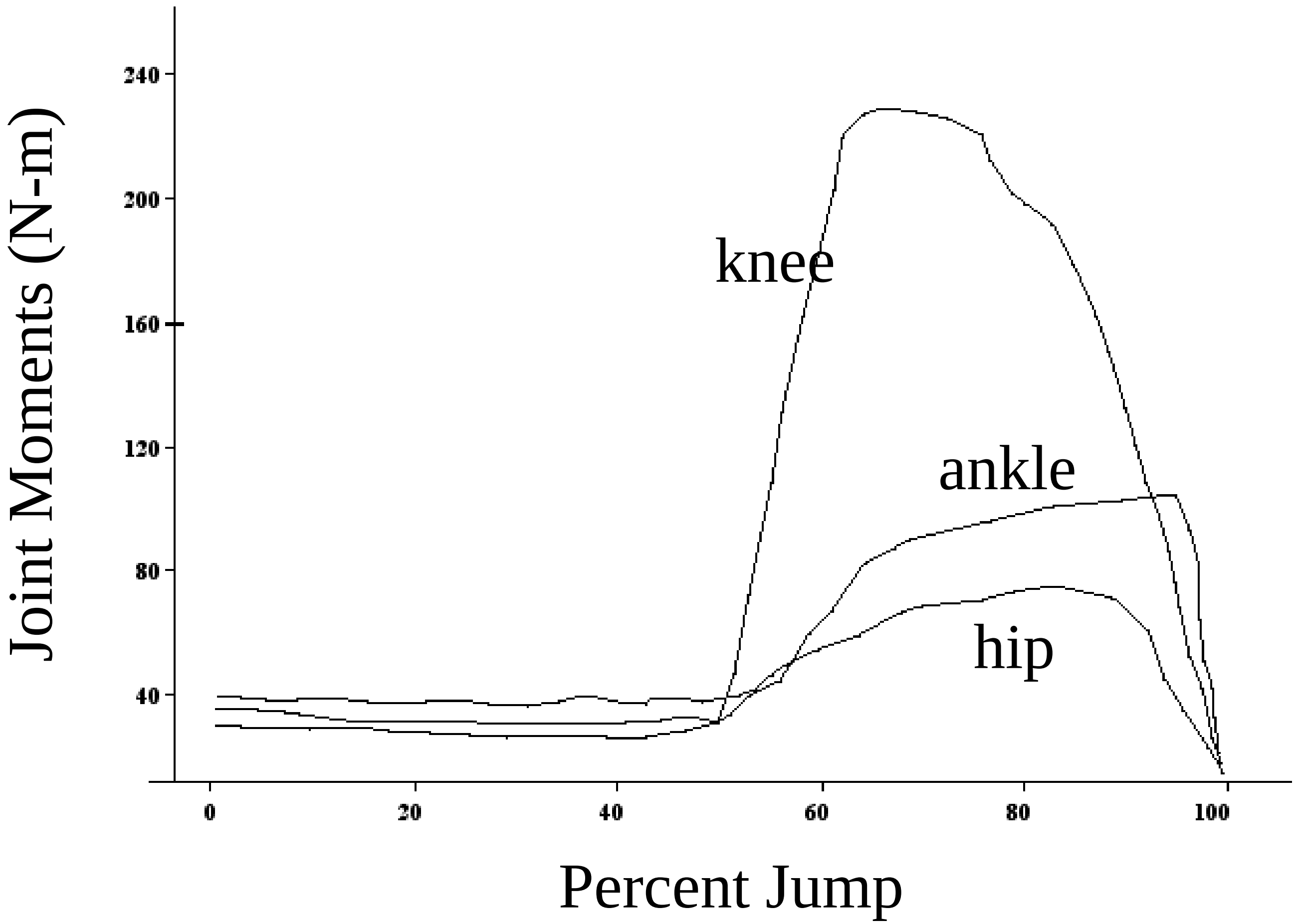
- segment masses
- segment lengths
- inertial properties

How do we get these values?



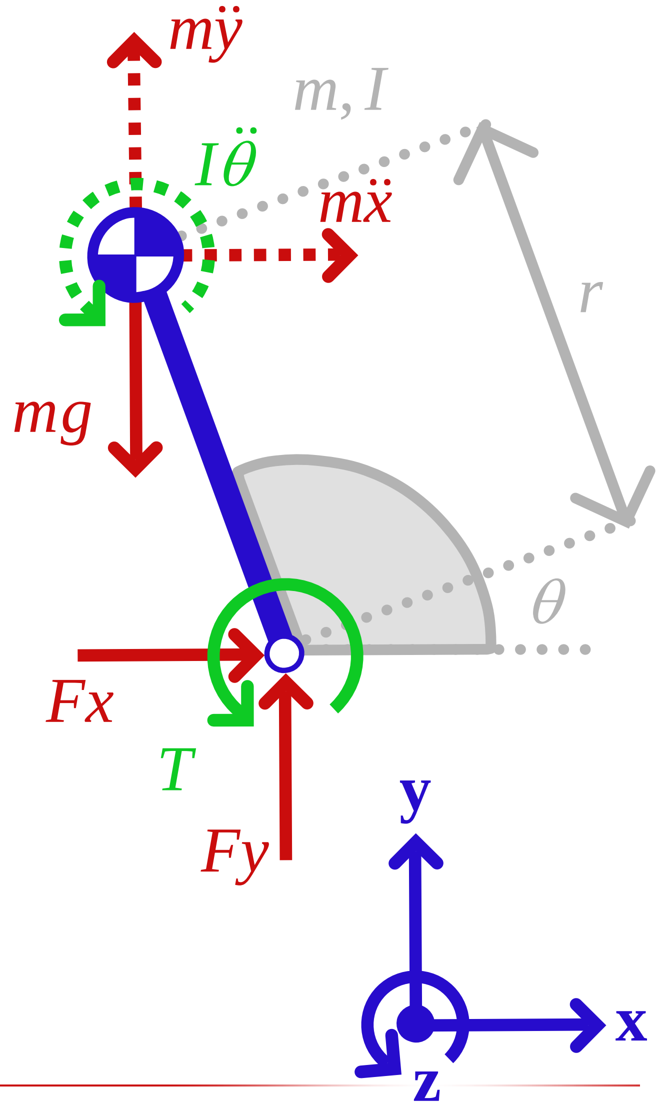
Zajac & Gordon, 1989

Use Inverse Dynamics to Calculate Net Joint Moments



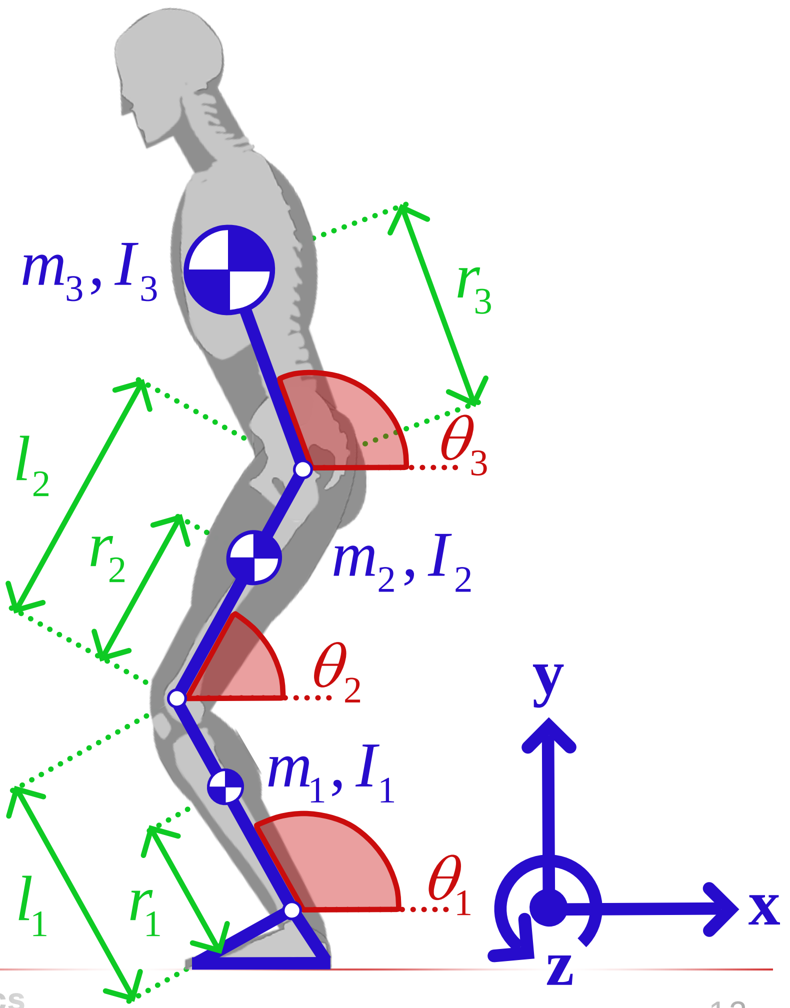
3 Steps to Inverse Dynamics

1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐ Inertia forces & moments
2. Form motion quantities (kinematics)
   - ☐ Positions $(x,\ y,\ \theta)$
   - ☐ Velocities $(\dot{x},\ \dot{y},\ \dot{\theta})$
   - ☐ Accelerations $(\ddot{x},\ \ddot{y},\ \ddot{\theta})$
3. Apply Newton's 2nd Law (kinetics)
   - ☐ Forces $(\Sigma F_x = m\ddot{x}),\ (\Sigma F_y = m\ddot{y})$
   - ☐ Moments $(\Sigma M = I\ddot{\theta})$



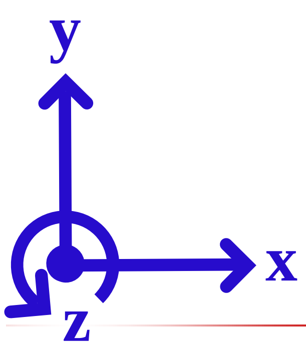
Example Problem

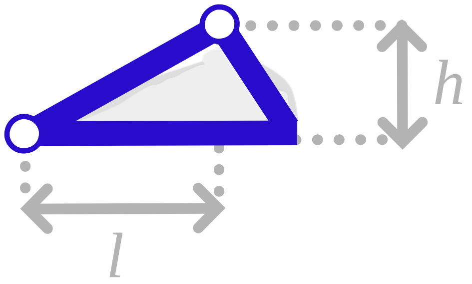
- Planar 3 degrees of freedom
- Position (orientation) in global coordinate system
- Segment length = l_i
- Distance to mass center = r_i
- Moments of inertia about mass center
- Foot has no mass and remains on ground



Segment 0 (foot) is Ground

1. Create free body diagram (statics *NOT* dynamics)
   - ☐ Reaction forces & moments





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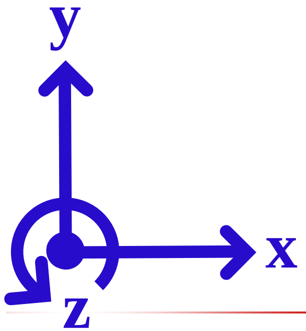

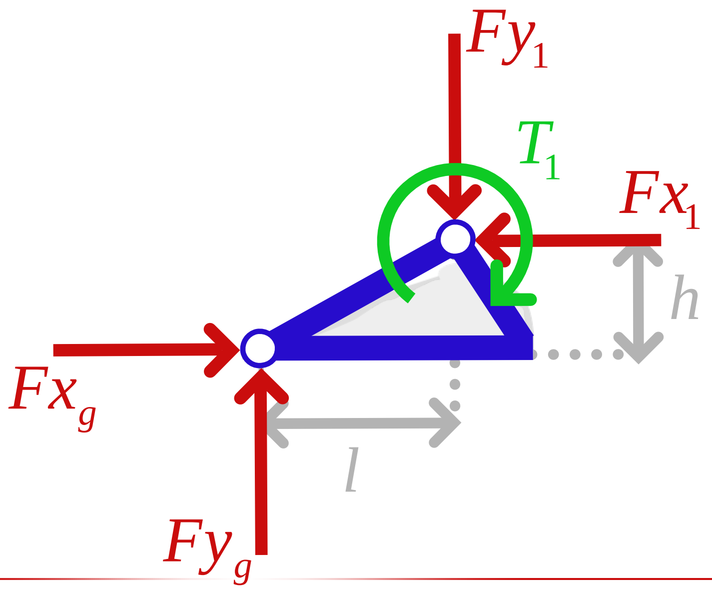

Segment 0 (foot) is Ground

1. Create free body diagram (statics *NOT* dynamics)
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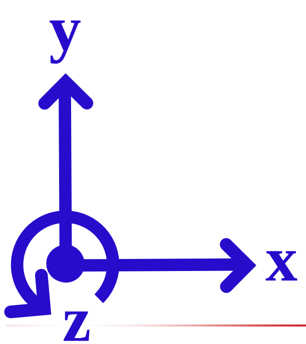
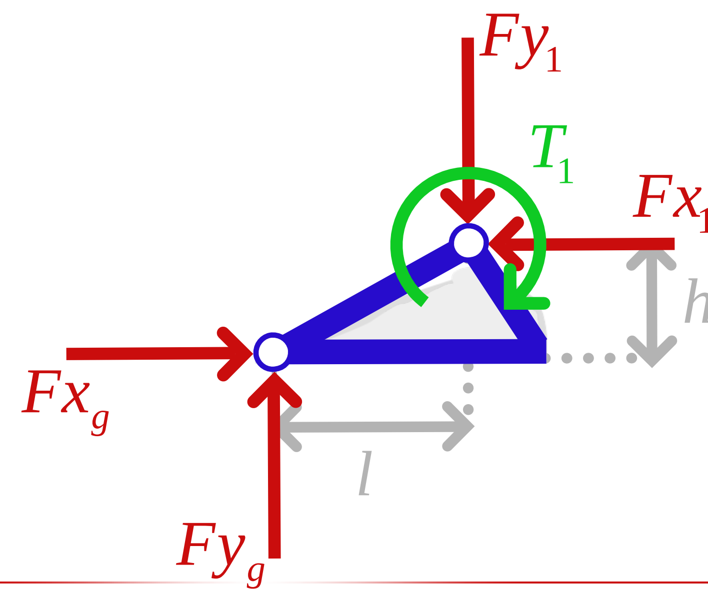
2. Apply Newton's 1st Law ($ma = 0$)
   - ☐ Forces
   - ☐ Moments





Segment 0 (foot) is Ground

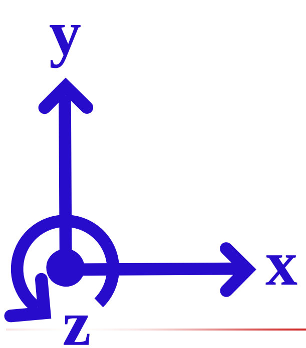
1. Create free body diagram (statics *NOT* dynamics)
   - Reaction forces & moments
2. Apply Newton's 1st Law (ma = 0)
   - Forces $\quad \Sigma Fx = 0 \quad \Sigma Fy = 0$

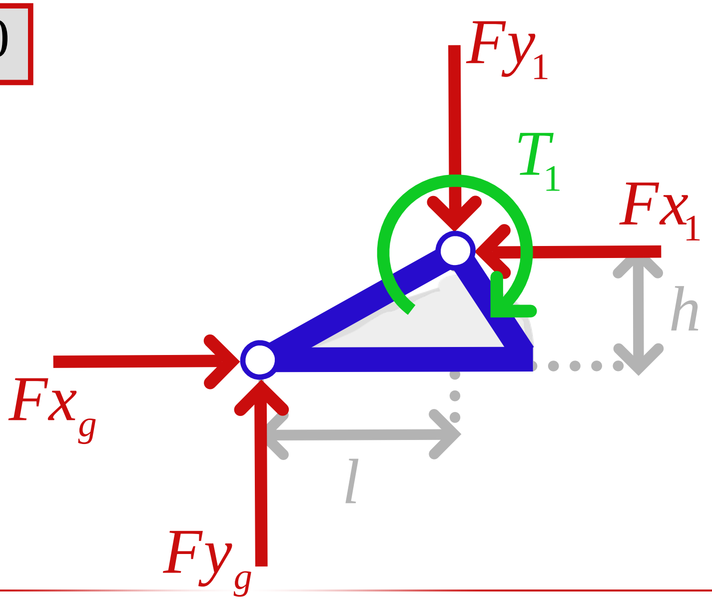
     $$\boxed{Fx_g - Fx_1 = 0} \qquad \boxed{Fy_g - Fy_1 = 0}$$

   - Moments $\quad \Sigma M = I_1\ddot{\theta}_1$

     $$\boxed{Fx_g h - Fy_g l - T_1 = 0}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} Fx_1 \\ Fy_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} Fx_g \\ Fy_g \\ Fx_g h - Fy_g l \end{bmatrix}$$





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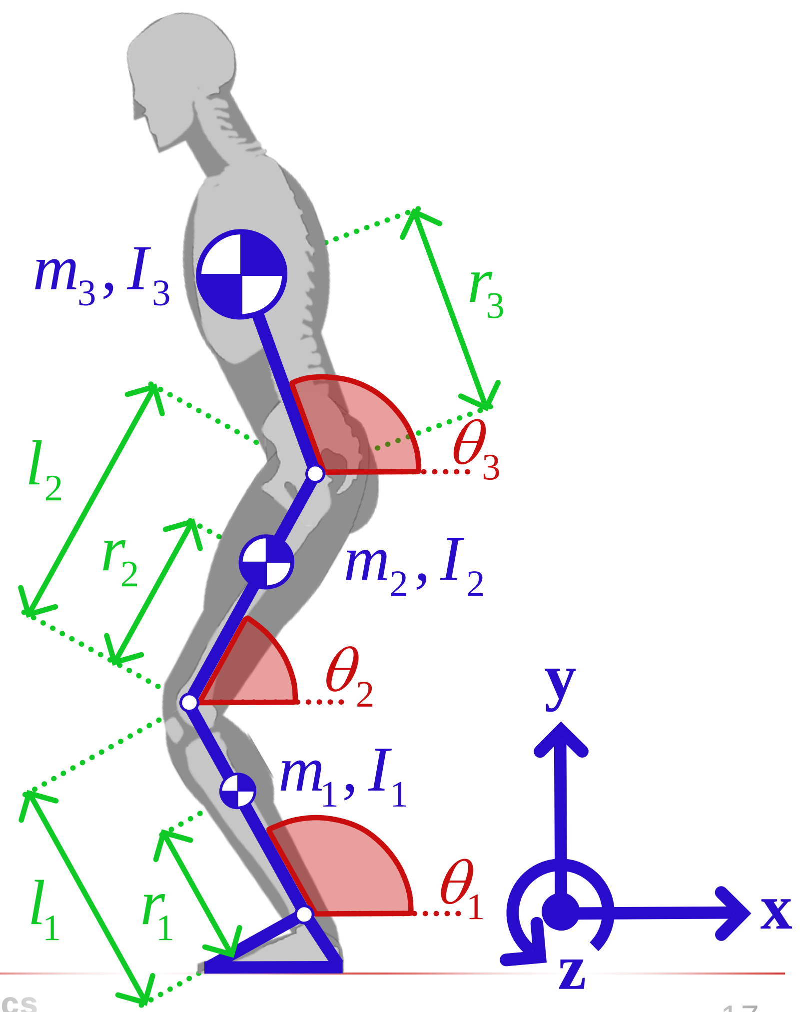
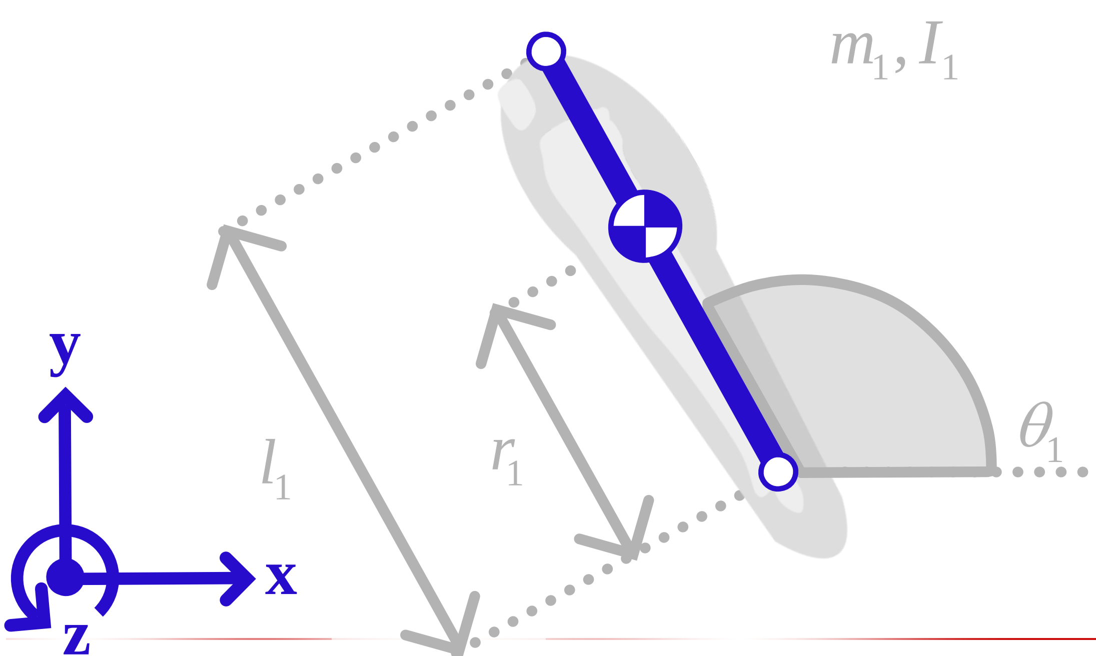


Segment 1 (shank)

1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐ Inertia forces & moments



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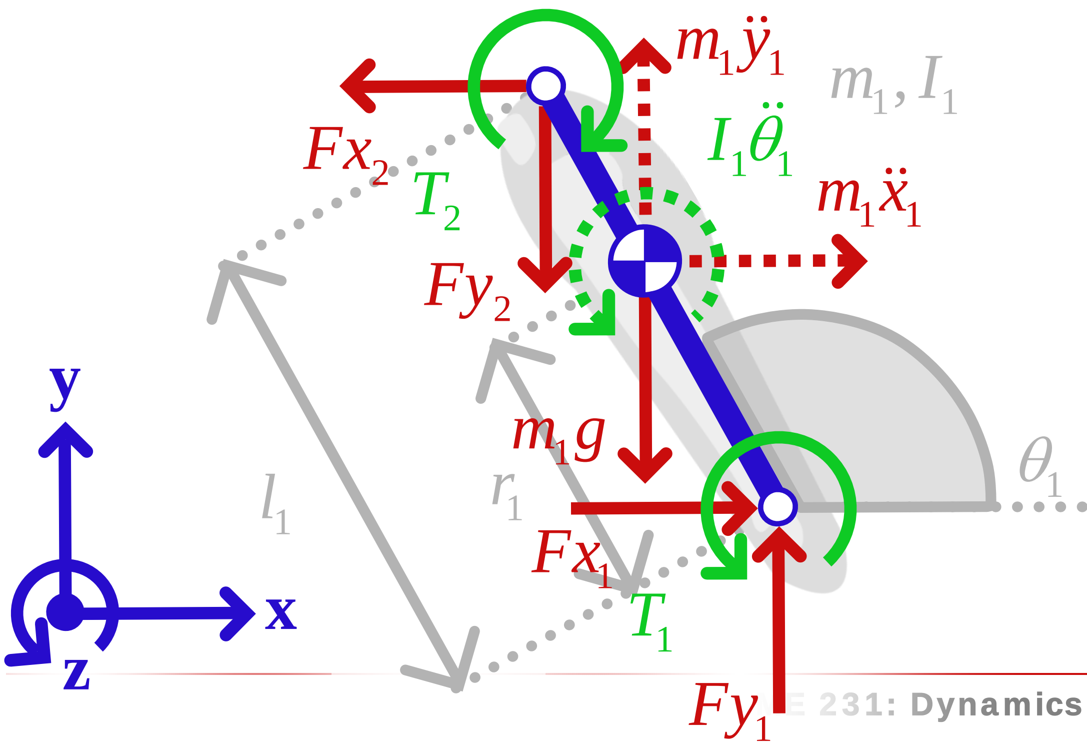
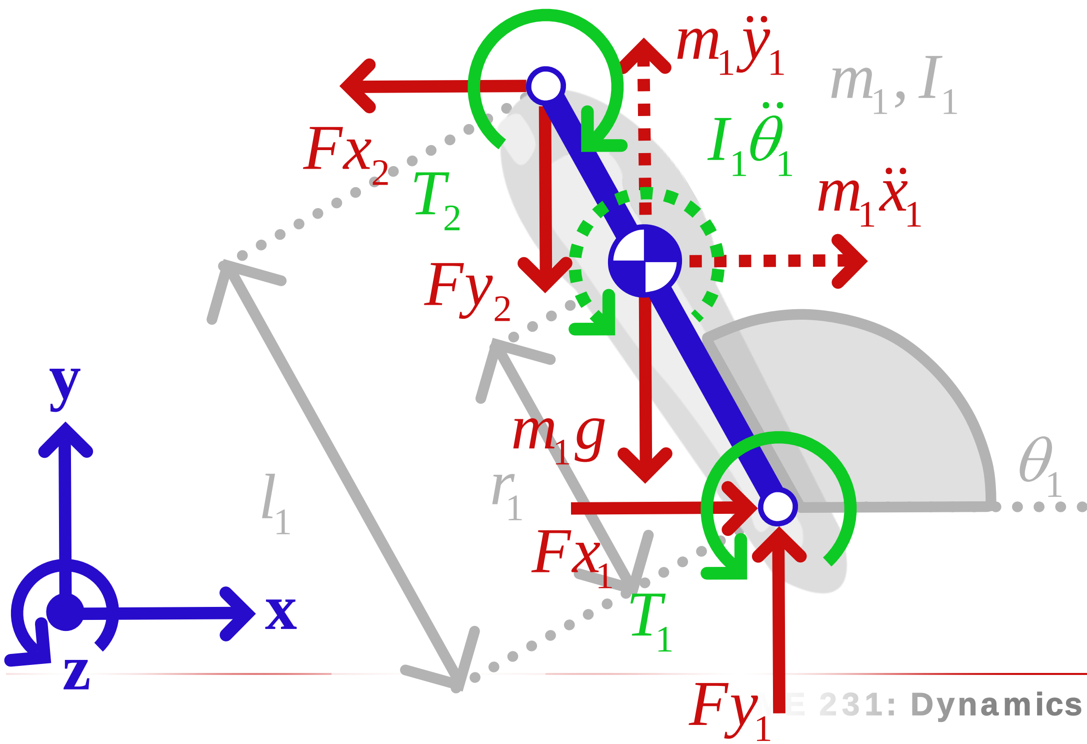
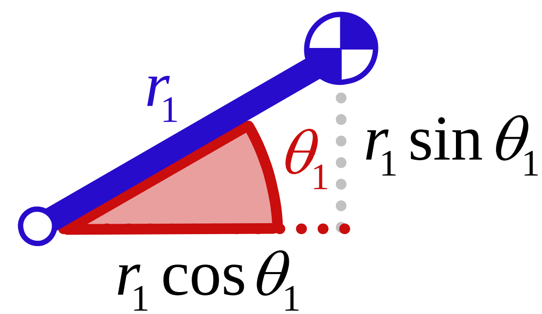


Segment 1 (shank)

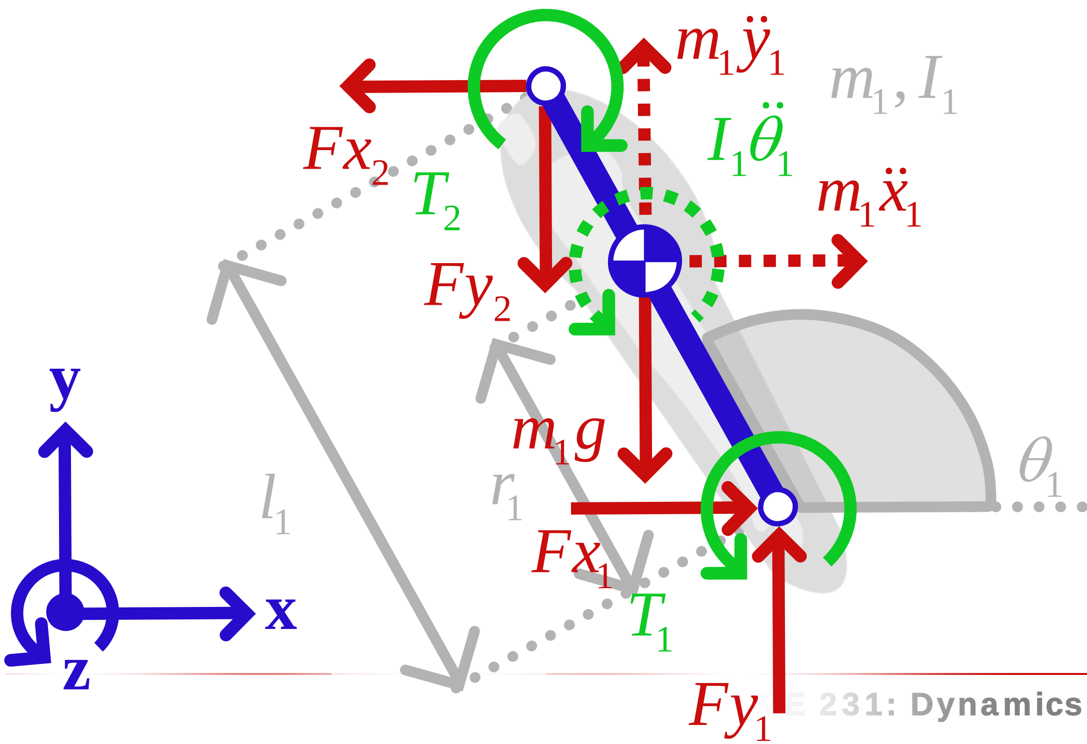
2. Form motion quantities (kinematics)
   - ☐ Positions
   - ☐ Velocities
   - ☐ Accelerations

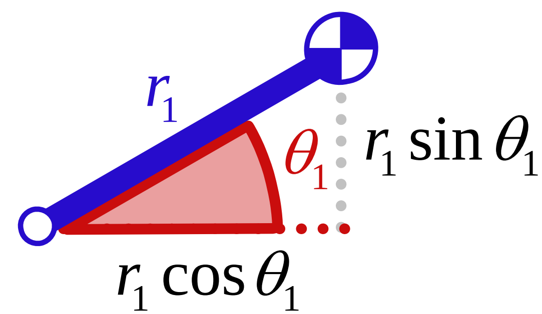




Segment 1 (shank)

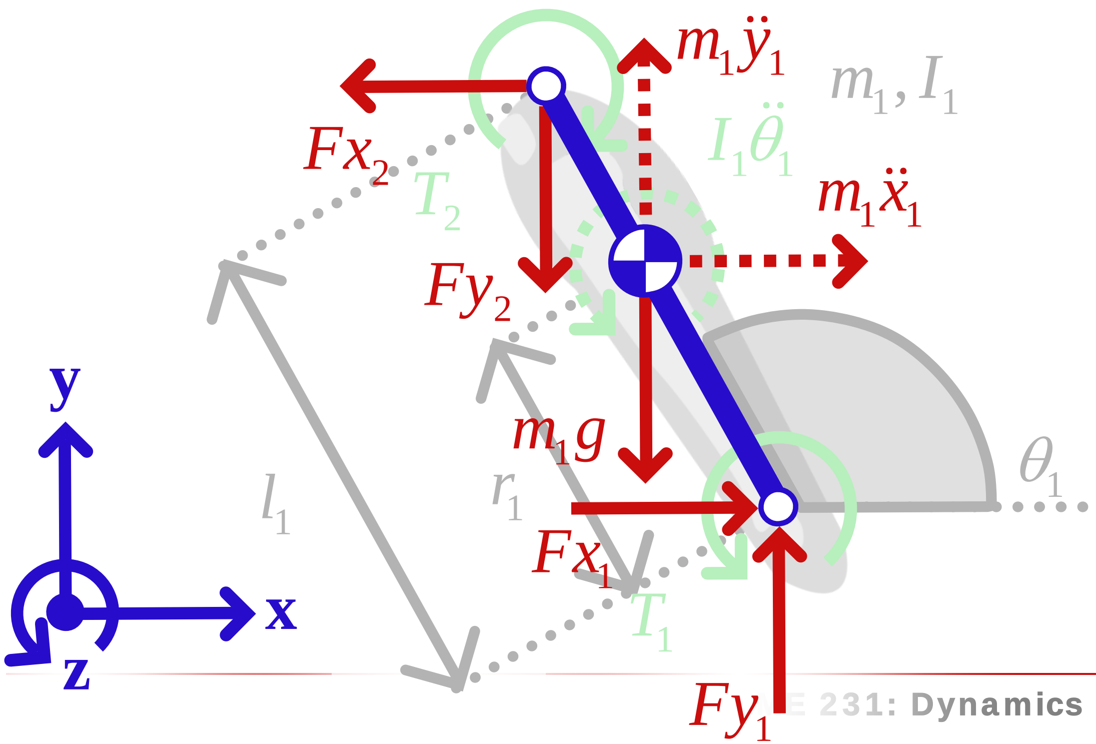
2. Form motion quantities (kinematics)
   - ☐ Positions $x_1 = r_1 c\theta_1$ $y_1 = r_1 s\theta_1$
   - ☐ Velocities $\dot{x}_1 = -r_1 s\theta_1 \dot{\theta}_1$ $\dot{y}_1 = r_1 c\theta_1 \dot{\theta}_1$
   - ☐ Accelerations $\ddot{x}_1 = -r_1\left(s\theta_1\ddot{\theta}_1 + c\theta_1\dot{\theta}_1^2\right)$ $\ddot{y}_1 = r_1\left(c\theta_1\ddot{\theta}_1 - s\theta_1\dot{\theta}_1^2\right)$





Segment 1 (shank)

3. Apply Newton's 2nd Law (kinetics)
   - Forces



Segment 1 (shank)

3. Apply Newton's 2nd Law (kinetics)
   - Forces

$$\Sigma Fx = m_1 \ddot{x}_1$$

$$Fx_1 - Fx_2 = m_1 \ddot{x}_1$$

1

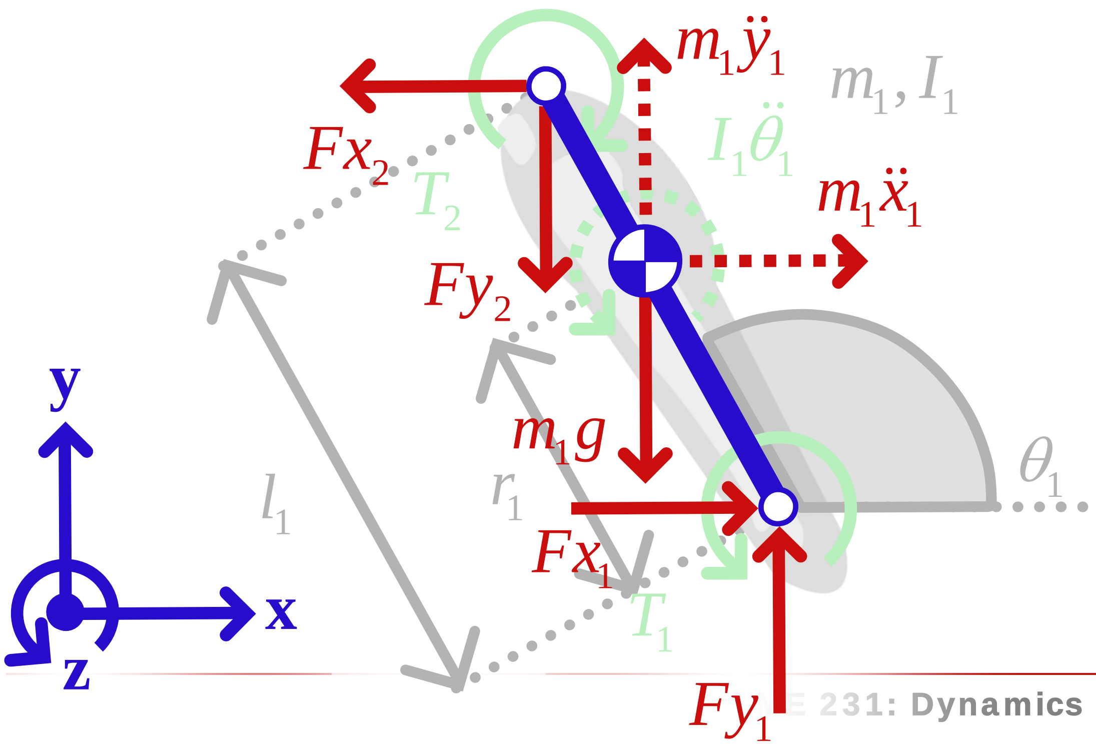
$$\boxed{Fx_1 - Fx_2 = -m_1 r_1 \left(s\theta_1 \ddot{\theta}_1 + c\theta_1 \dot{\theta}_1^2 \right)}$$

$$\Sigma Fy = m_1 \ddot{y}_1$$

$$Fy_1 - Fy_2 - m_1 g = m_1 \ddot{y}_1$$

2

$$\boxed{Fy_1 - Fy_2 - m_1 g = m_1 r_1 \left(c\theta_1 \ddot{\theta}_1 - s\theta_1 \dot{\theta}_1^2 \right)}$$



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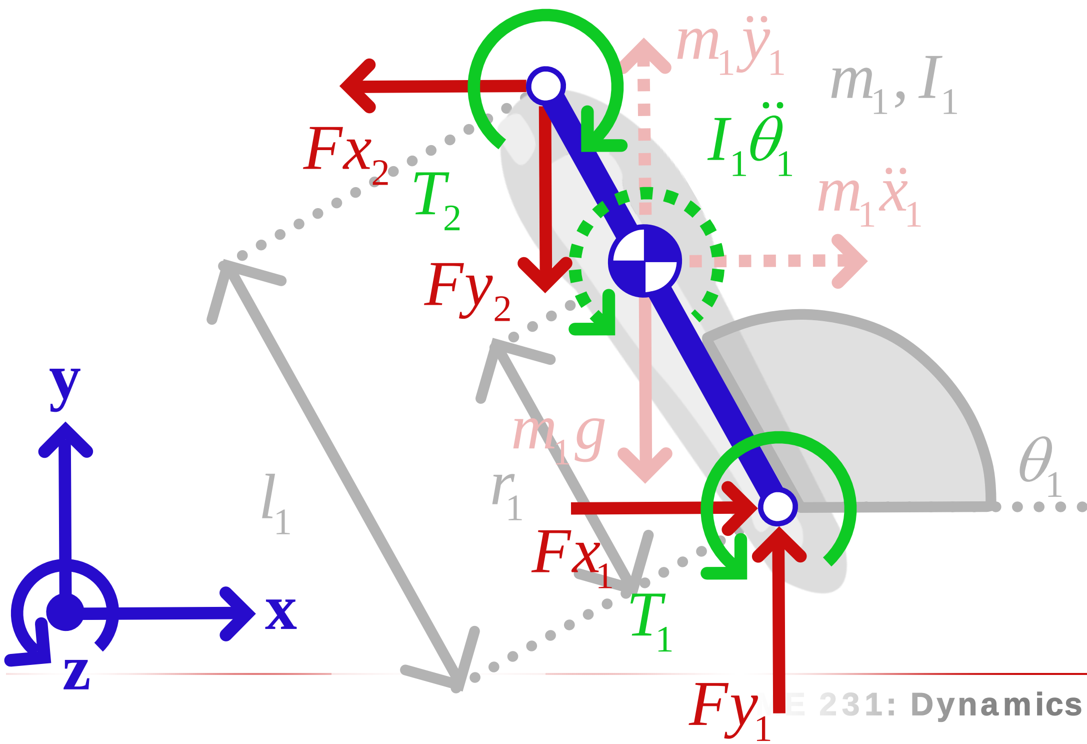
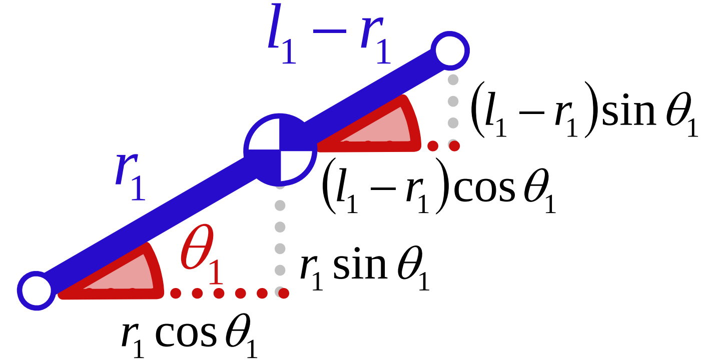
2 $$\boxed{Fy_1 - Fy_2 - m_1 g = m_1 r_1 \left(c\theta_1 \ddot{\theta}_1 - s\theta_1 \dot{\theta}_1^2 \right)}$$

$$\begin{array}{c} 1 \\ 2 \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} Fx_1 \\ Fx_2 \\ Fy_1 \\ Fy_2 \end{bmatrix} = \begin{bmatrix} -m_1 r_1 \left(s\theta_1 \ddot{\theta}_1 + c\theta_1 \dot{\theta}_1^2 \right) \\ m_1 \left(r_1 \left(c\theta_1 \ddot{\theta}_1 - s\theta_1 \dot{\theta}_1^2 \right) + g \right) \end{bmatrix}$$

Segment 1 (shank)

3. Apply Newton's 2nd Law (kinetics)
   - ☐ Moments





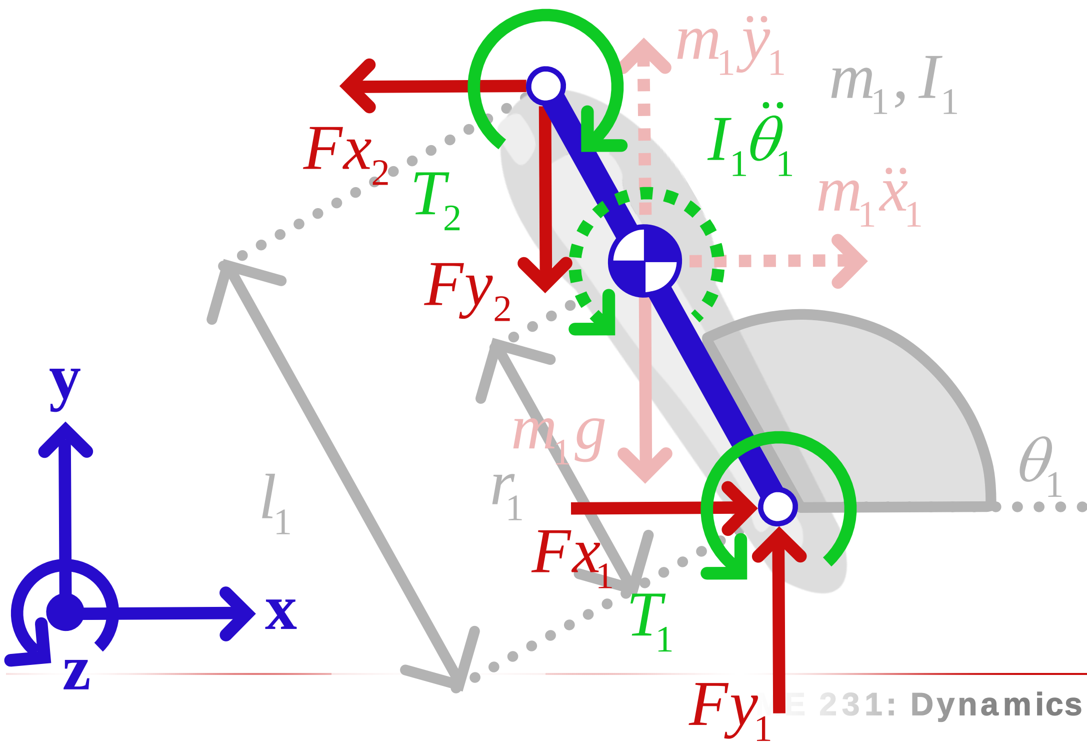
Segment 1 (shank)

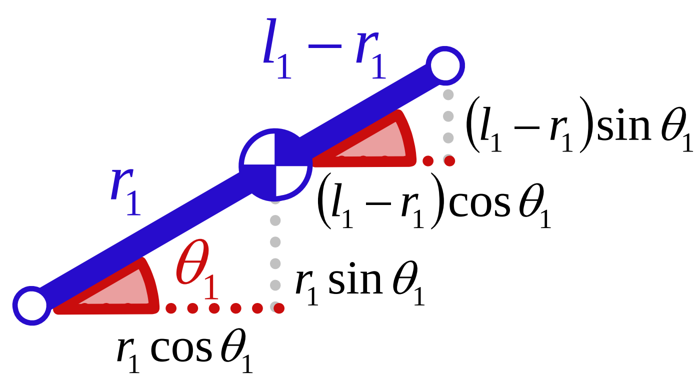
3. Apply Newton's 2nd Law (kinetics)
   - ☐ Moments

$$\Sigma M = I_1\ddot{\theta}_1$$

3

$$\boxed{T_1 - T_2 + Fx_1 r_1 s\theta_1 - Fy_1 r_1 c\theta_1 + Fx_2\left(l_1 - r_1\right)s\theta_1 - Fy_2\left(l_1 - r_1\right)c\theta_1 = I_1\ddot{\theta}_1}$$





Segment 1 (shank)

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   - Moments

$$\Sigma M = I_1\ddot{\theta}_1$$

$$\boxed{T_1 - T_2 + Fx_1 r_1 s\theta_1 - Fy_1 r_1 c\theta_1 + Fx_2(l_1 - r_1)s\theta_1 - Fy_2(l_1 - r_1)c\theta_1 = I_1\ddot{\theta}_1}$$ 3

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & r_1 s\theta_1 & -r_1 c\theta_1 & (l_1 - r_1)s\theta_1 & (l_1 - r_1)c\theta_1 \end{bmatrix}\begin{bmatrix} Fx_1 \\ Fx_2 \\ Fy_1 \\ Fy_2 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -m_1 r_1\left(s\theta_1\ddot{\theta}_1 + c\theta_1\dot{\theta}_1^2\right) \\ m_1\left(r_1\left(c\theta_1\ddot{\theta}_1 - s\theta_1\dot{\theta}_1^2\right) + g\right) \\ I_1\ddot{\theta}_1 \end{bmatrix}$$

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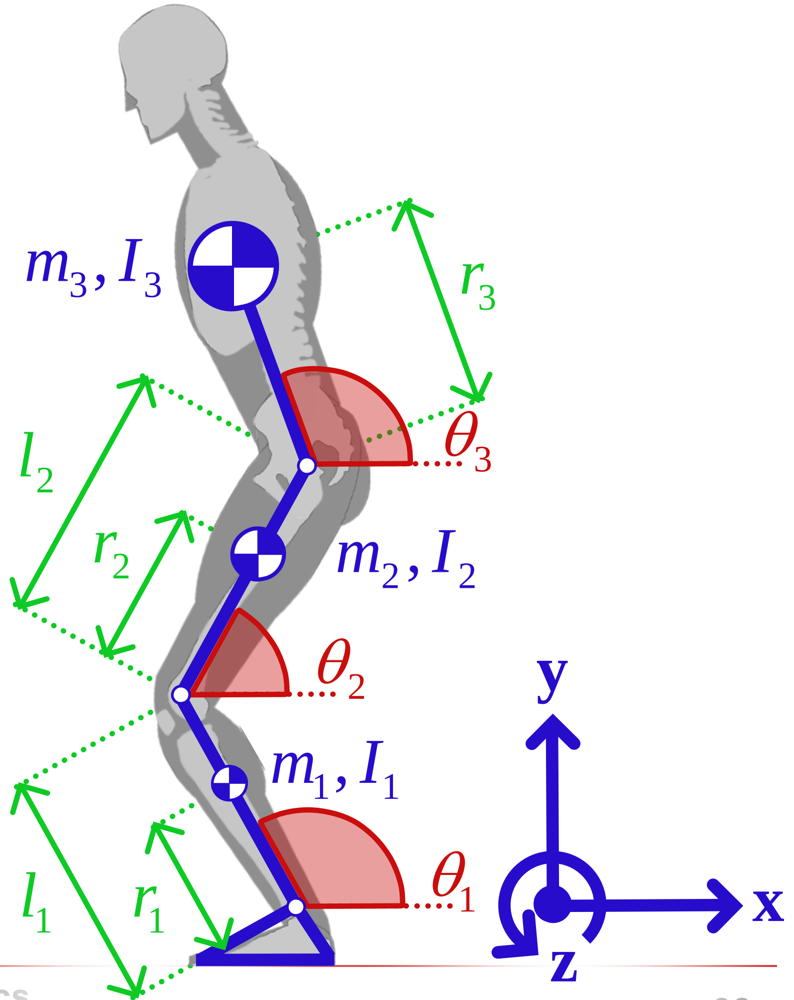
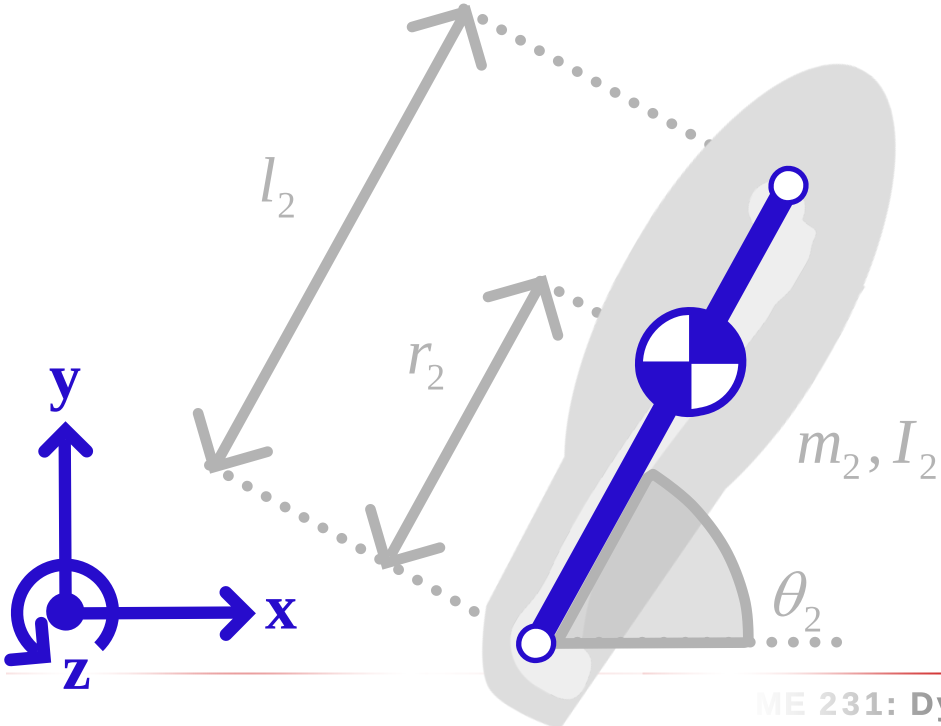


Segment 2 (thigh)

1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐ Inertia forces & moments



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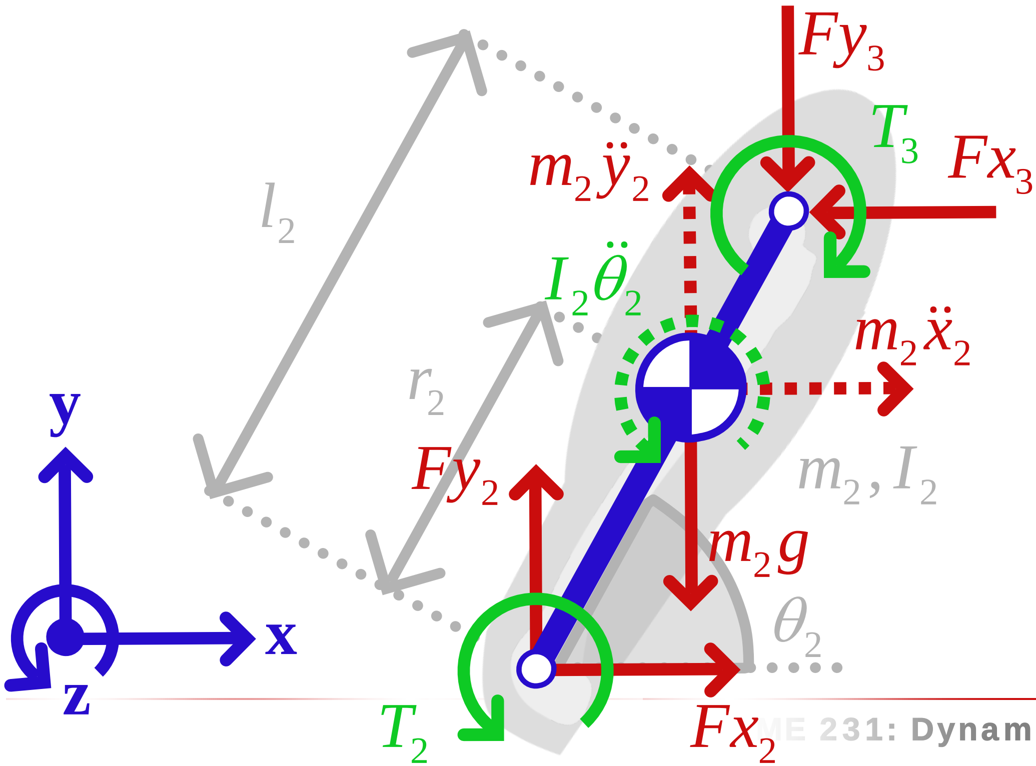
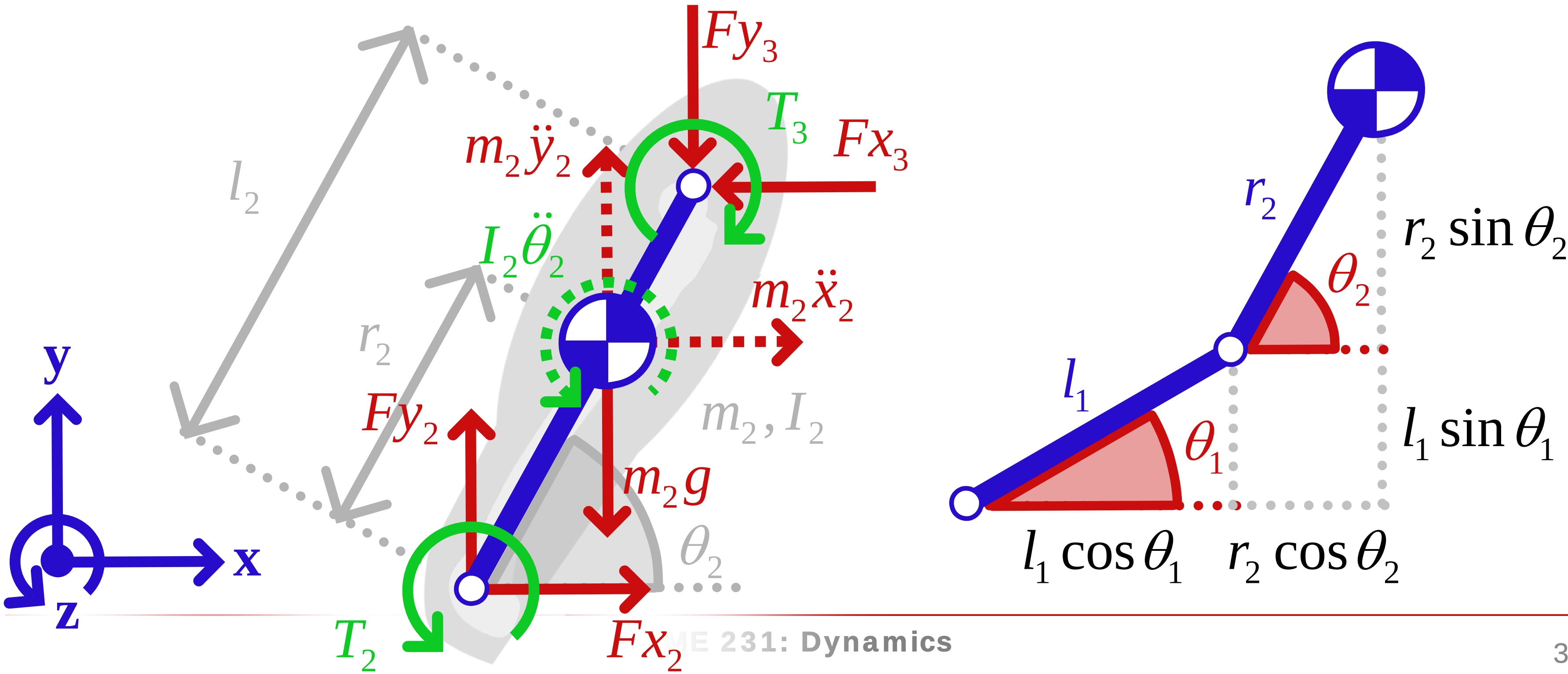


Segment 2 (thigh)

2. Form motion quantities (kinematics)
   - ☐ Positions
   - ☐ Velocities
   - ☐ Accelerations



Segment 2 (thigh)

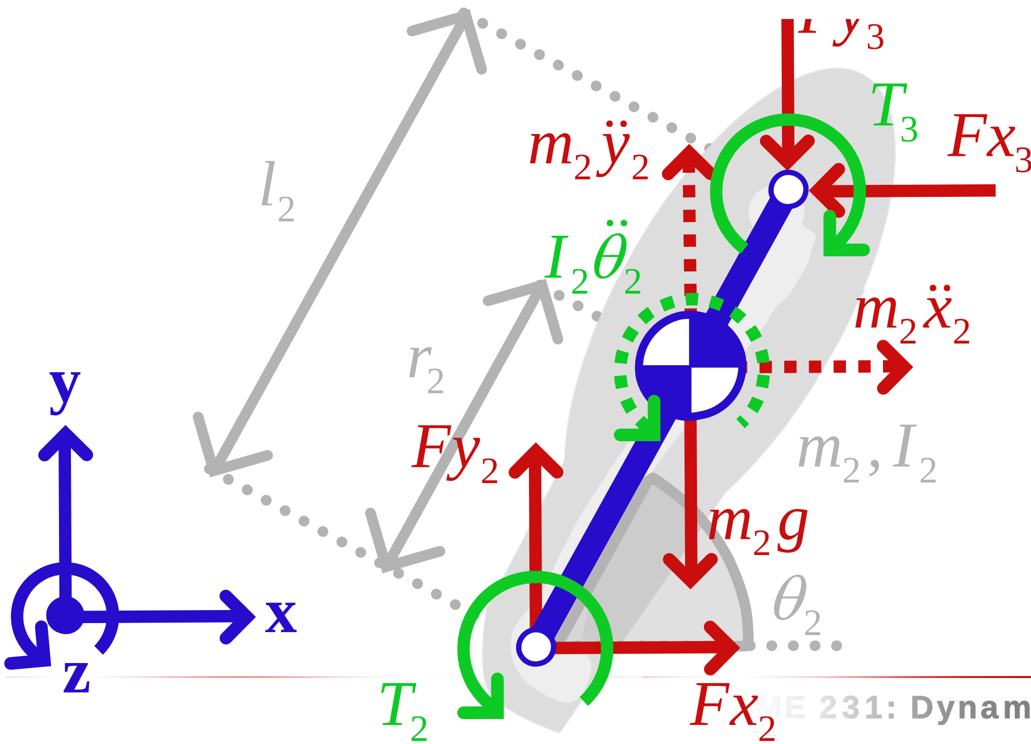
2. Form motion quantities (kinematics)
   - ☐ Positions
   - ☐ Velocities
   - ☐ Accelerations

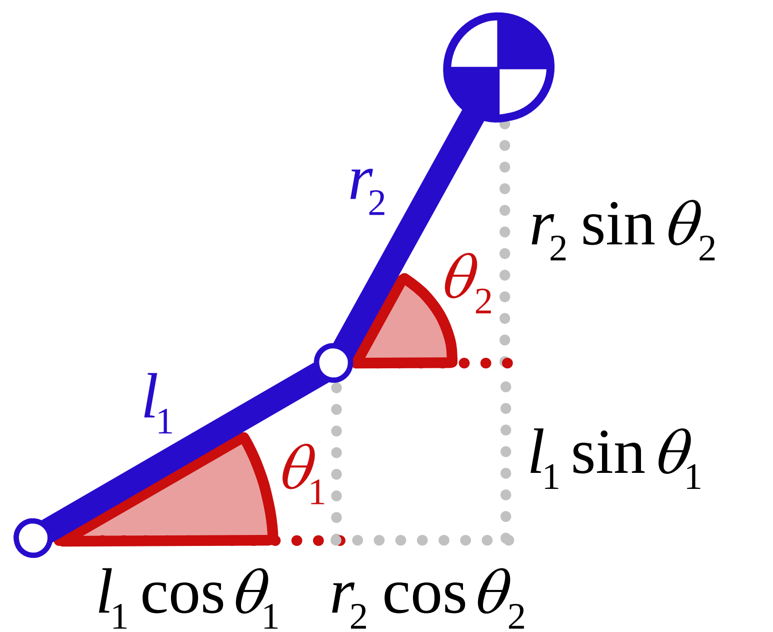
$$x_2 = l_1 \mathrm{c}\theta_1 + r_2 \mathrm{c}\theta_2 \qquad y_2 = l_1 \mathrm{s}\theta_1 + r_2 \mathrm{s}\theta_2$$

$$\dot{x}_2 = -l_1 \mathrm{s}\theta_1 \dot{\theta}_1 - r_2 \mathrm{s}\theta_2 \dot{\theta}_2 \qquad \dot{y}_2 = l_1 \mathrm{c}\theta_1 \dot{\theta}_1 + r_2 \mathrm{c}\theta_2 \dot{\theta}_2$$

$$\ddot{x}_2 = -l_1\left(\mathrm{s}\theta_1 \ddot{\theta}_1 + \mathrm{c}\theta_1 \dot{\theta}_1^2\right) - r_2\left(\mathrm{s}\theta_2 \ddot{\theta}_2 + \mathrm{c}\theta_2 \dot{\theta}_2^2\right)$$

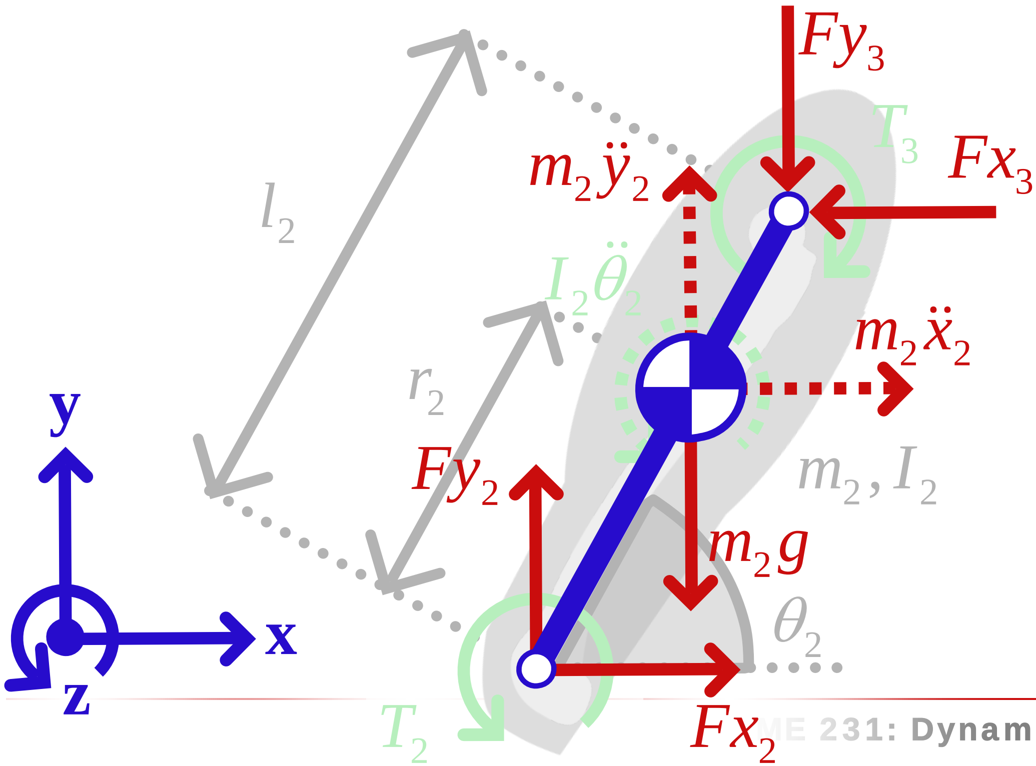
$$\ddot{y}_2 = l_1\left(\mathrm{c}\theta_1 \ddot{\theta}_1 - \mathrm{s}\theta_1 \dot{\theta}_1^2\right) + r_2\left(\mathrm{c}\theta_2 \ddot{\theta}_2 - \mathrm{s}\theta_2 \dot{\theta}_2^2\right)$$





Segment 2 (thigh)

3. Apply Newton's 2nd Law (kinetics)
   - Forces



Segment 2 (thigh)

3. Apply Newton's 2nd Law (kinetics)
   - ☐ Forces

$$\Sigma Fx = m_2 \ddot{x}_2$$

$$Fx_2 - Fx_3 = m_2 \ddot{x}_2$$

4

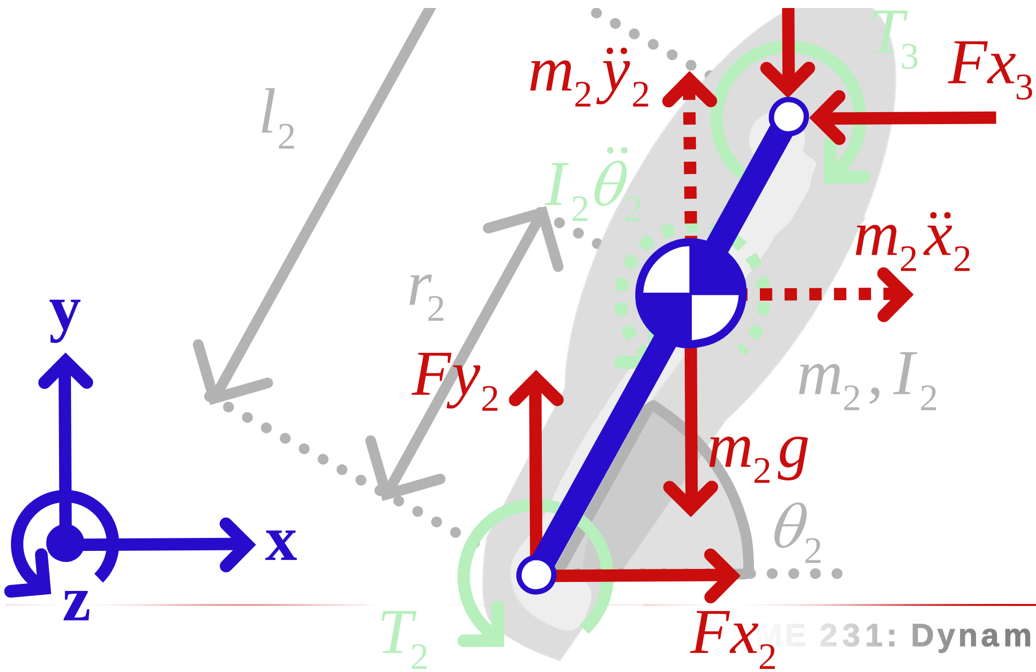
$$\boxed{Fx_2 - Fx_3 = -m_2 \begin{pmatrix} l_1\left(s\theta_1\ddot{\theta}_1 + c\theta_1\dot{\theta}_1^2\right) \\ + r_2\left(s\theta_2\ddot{\theta}_2 + c\theta_2\dot{\theta}_2^2\right) \end{pmatrix}}$$

$$\Sigma Fy = m_2 \ddot{y}_2$$

$$Fy_2 - Fy_3 - m_2 g = m_2 \ddot{y}_2$$

5

$$\boxed{\begin{pmatrix} Fy_2 - Fy_3 \\ - m_2 g \end{pmatrix} = m_2 \begin{pmatrix} l_1\left(c\theta_1\ddot{\theta}_1 - s\theta_1\dot{\theta}_1^2\right) \\ + r_2\left(c\theta_2\ddot{\theta}_2 - s\theta_2\dot{\theta}_2^2\right) \end{pmatrix}}$$



Segment 2 (thigh)

3. Apply Newton's 2[nd] Law (kinetics)
   - ☐ Forces

$$\Sigma Fx = m_2 \ddot{x}_2$$

$$Fx_2 - Fx_3 = m_2 \ddot{x}_2$$

4

$$\boxed{Fx_2 - Fx_3 = -m_2 \begin{pmatrix} l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) \\ +\, r_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right) \end{pmatrix}}$$

$$\Sigma Fy = m_2 \ddot{y}_2$$

$$Fy_2 - Fy_3 - m_2 g = m_2 \ddot{y}_2$$

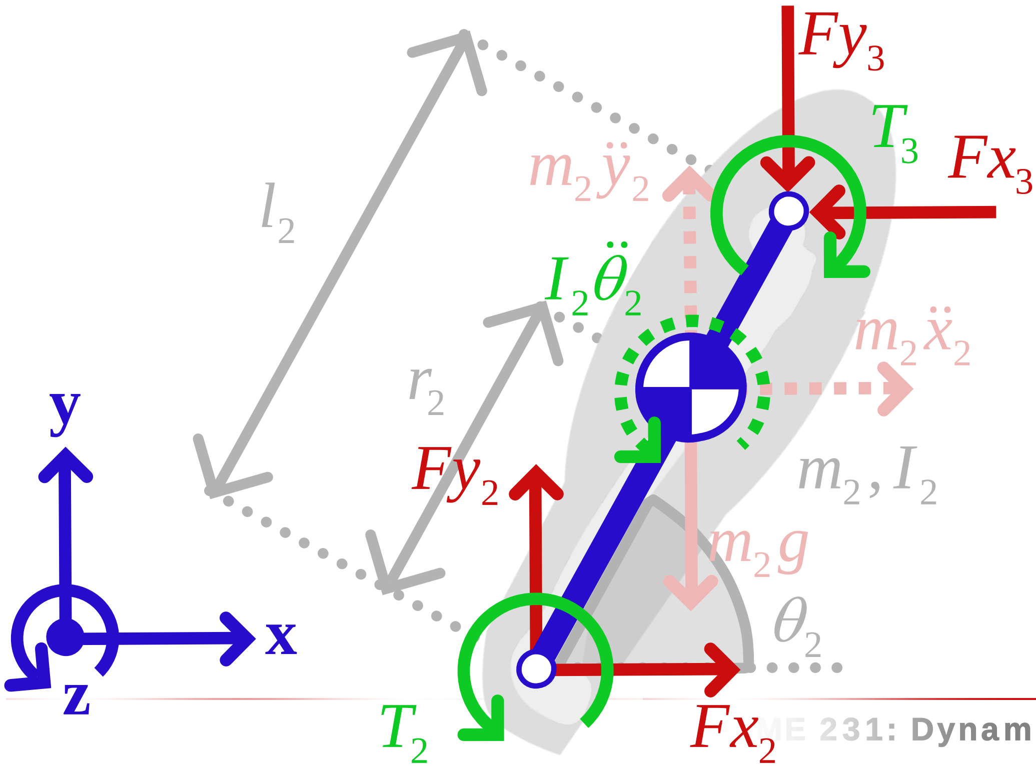
5

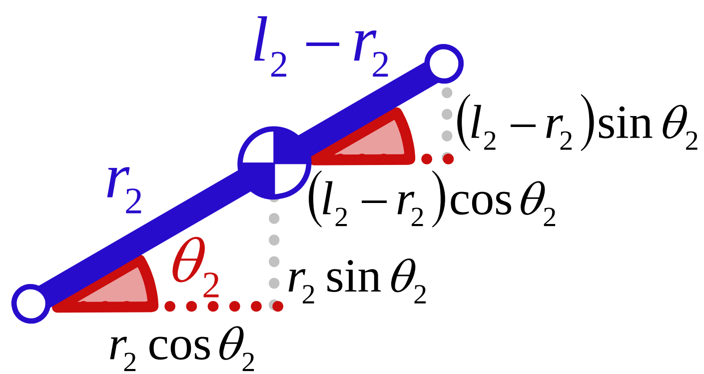
$$\boxed{\begin{pmatrix} Fy_2 - Fy_3 \\ -\, m_2 g \end{pmatrix} = m_2 \begin{pmatrix} l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) \\ +\, r_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) \end{pmatrix}}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & r_1\mathrm{s}\theta_1 & -r_1\mathrm{c}\theta_1 & (l_1 - r_1)\mathrm{s}\theta_1 & (l_1 - r_1)\mathrm{c}\theta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} Fx_1 \\ Fx_2 \\ Fy_1 \\ Fy_2 \\ T_1 \\ T_2 \\ Fx_3 \\ Fy_3 \end{bmatrix} = \begin{bmatrix} -m_1 r_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) \\ m_1\left(r_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + g\right) \\ I_1\ddot{\theta}_1 \\ -m_2\left(l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right)\right) \\ m_2\left(l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) + g\right) \end{bmatrix}$$

Segment 2 (thigh)

3. Apply Newton's 2nd Law (kinetics)
   - Moments





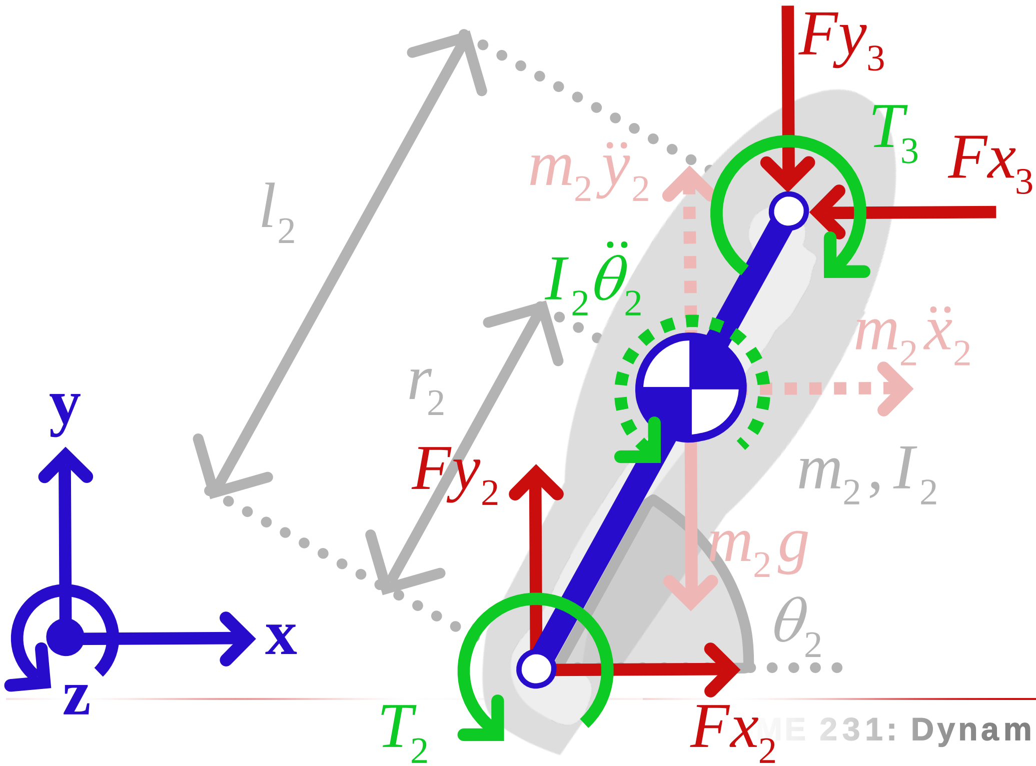
Segment 2 (thigh)

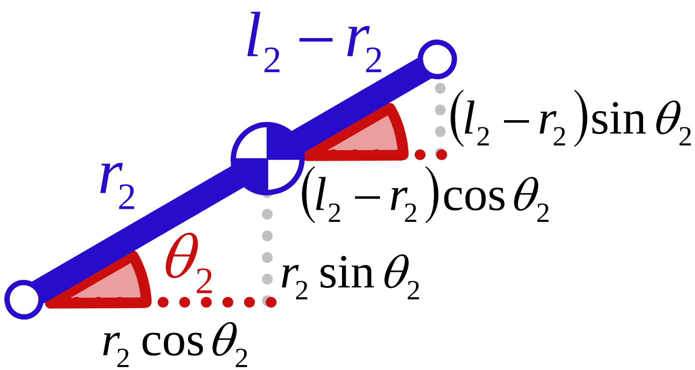
3. Apply Newton's 2nd Law (kinetics)
   - Moments

$$\Sigma M = I_2\ddot{\theta}_2$$

$$\boxed{T_2 - T_3 + Fx_2 r_2 s\theta_2 - Fy_2 r_2 c\theta_2 + Fx_3(l_2 - r_2)s\theta_2 - Fy_3(l_2 - r_2)c\theta_2 = I_2\ddot{\theta}_2}$$

6





Segment 2 (thigh)

3. Apply Newton's 2nd Law (kinetics)
   - Moments

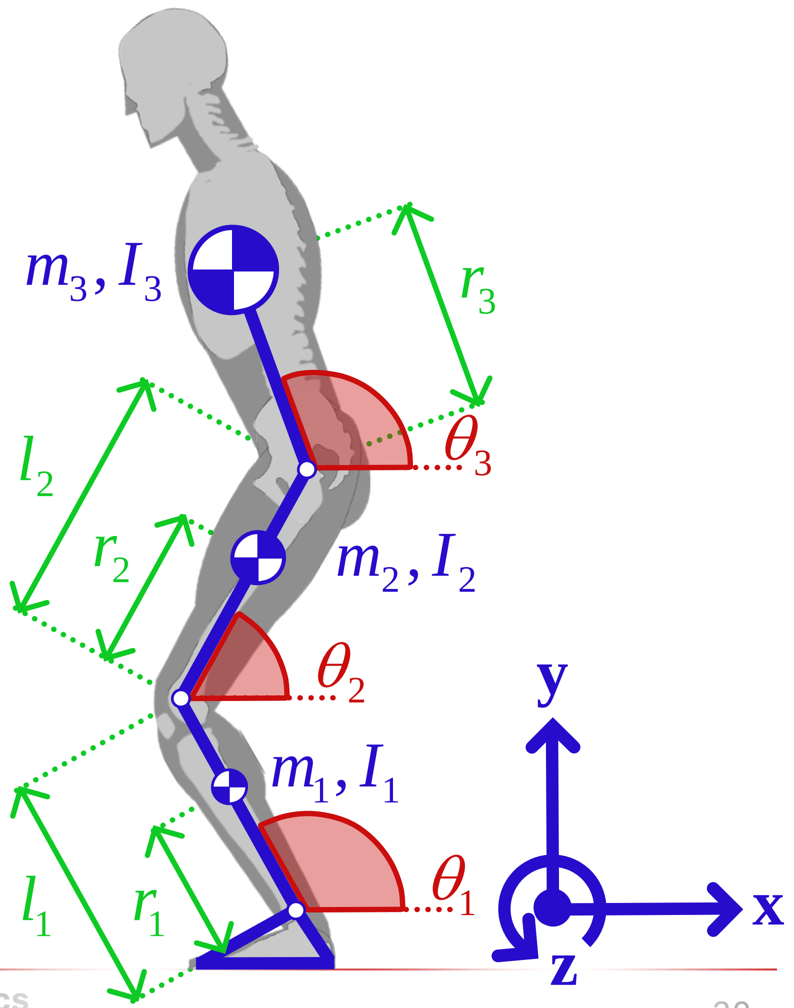
$$\Sigma M = I_2\ddot{\theta}_2$$

$$\boxed{T_2 - T_3 + Fx_2 r_2 \mathrm{s}\theta_2 - Fy_2 r_2 \mathrm{c}\theta_2 + Fx_3(l_2 - r_2)\mathrm{s}\theta_2 - Fy_3(l_2 - r_2)\mathrm{c}\theta_2 = I_2\ddot{\theta}_2} \tag{6}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & r_1 \mathrm{s}\theta_1 & -r_1 \mathrm{c}\theta_1 & (l_1 - r_1)\mathrm{s}\theta_1 & (l_1 - r_1)\mathrm{c}\theta_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & r_2 \mathrm{s}\theta_2 & 0 & -r_2 \mathrm{c}\theta_2 & 0 & 1 & (l_2 - r_2)\mathrm{s}\theta_2 & -(l_2 - r_2)\mathrm{c}\theta_2 & -1
\end{bmatrix}
\begin{bmatrix} Fx_1 \\ Fx_2 \\ Fy_1 \\ Fy_2 \\ T_1 \\ T_2 \\ Fx_3 \\ Fy_3 \\ T_3 \end{bmatrix}
=
\begin{bmatrix}
-m_1 r_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) \\
m_1\left(r_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + g\right) \\
I_1\ddot{\theta}_1 \\
-m_2\left(l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right)\right) \\
m_2\left(l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) + g\right) \\
I_2\ddot{\theta}_2
\end{bmatrix}$$

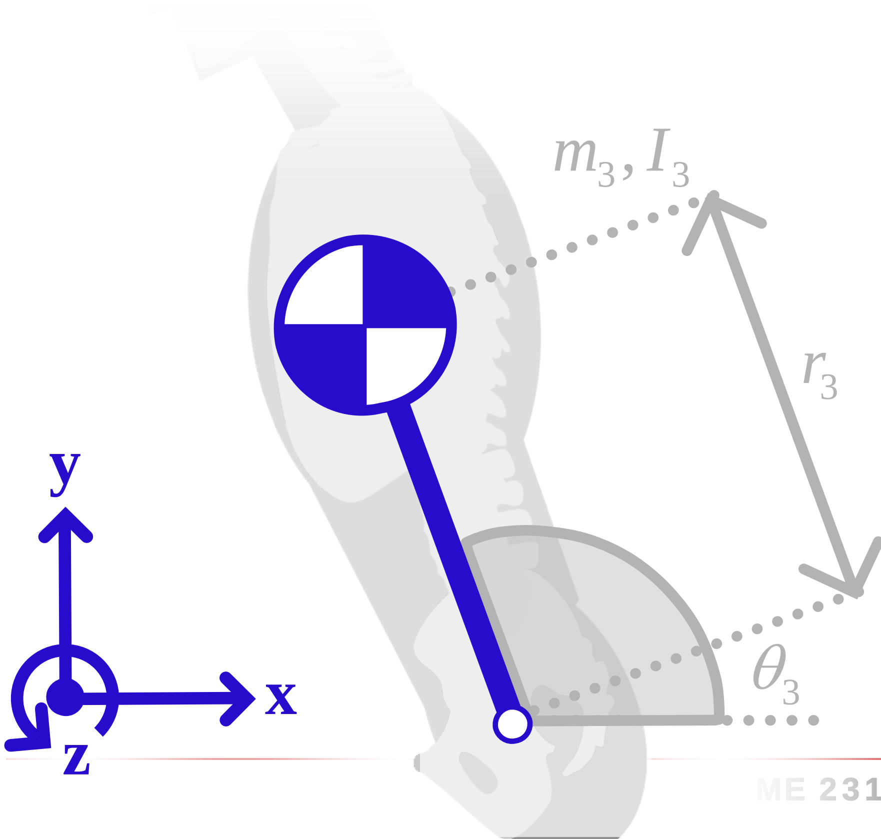
Example Problem

- Planar 3 degrees of freedom
- Position (orientation) in global coordinate system
- Segment length = l_i
- Distance to mass center = r_i
- Moments of inertia about mass center
- Foot has no mass and remains on ground



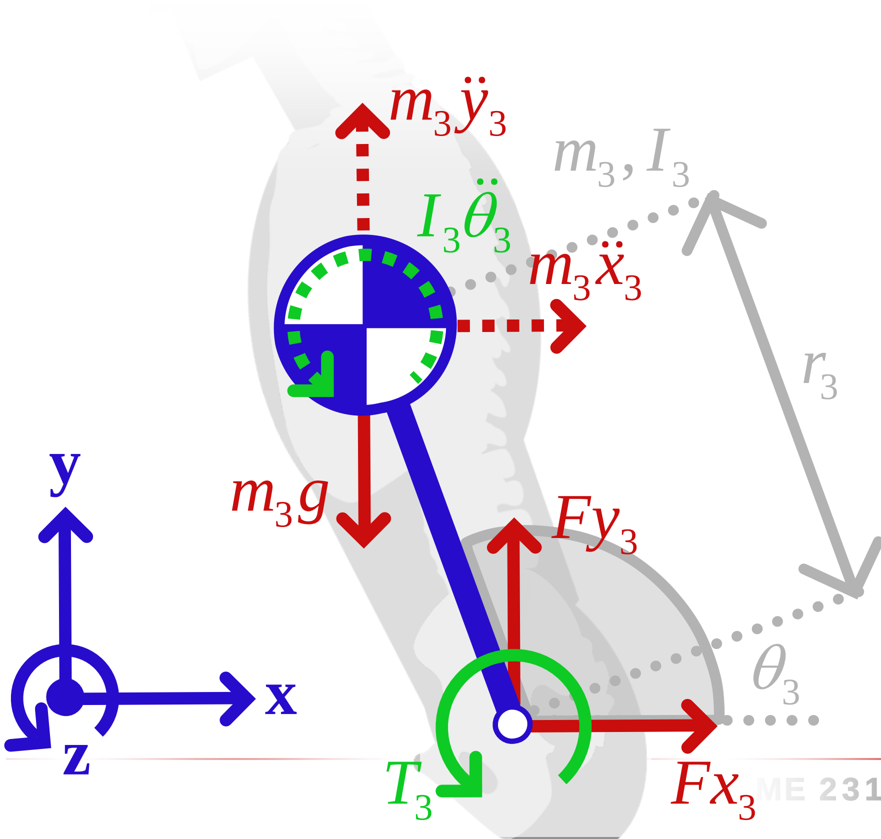
Segment 3 (head, arms, & trunk)

1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐ Inertia forces & moments



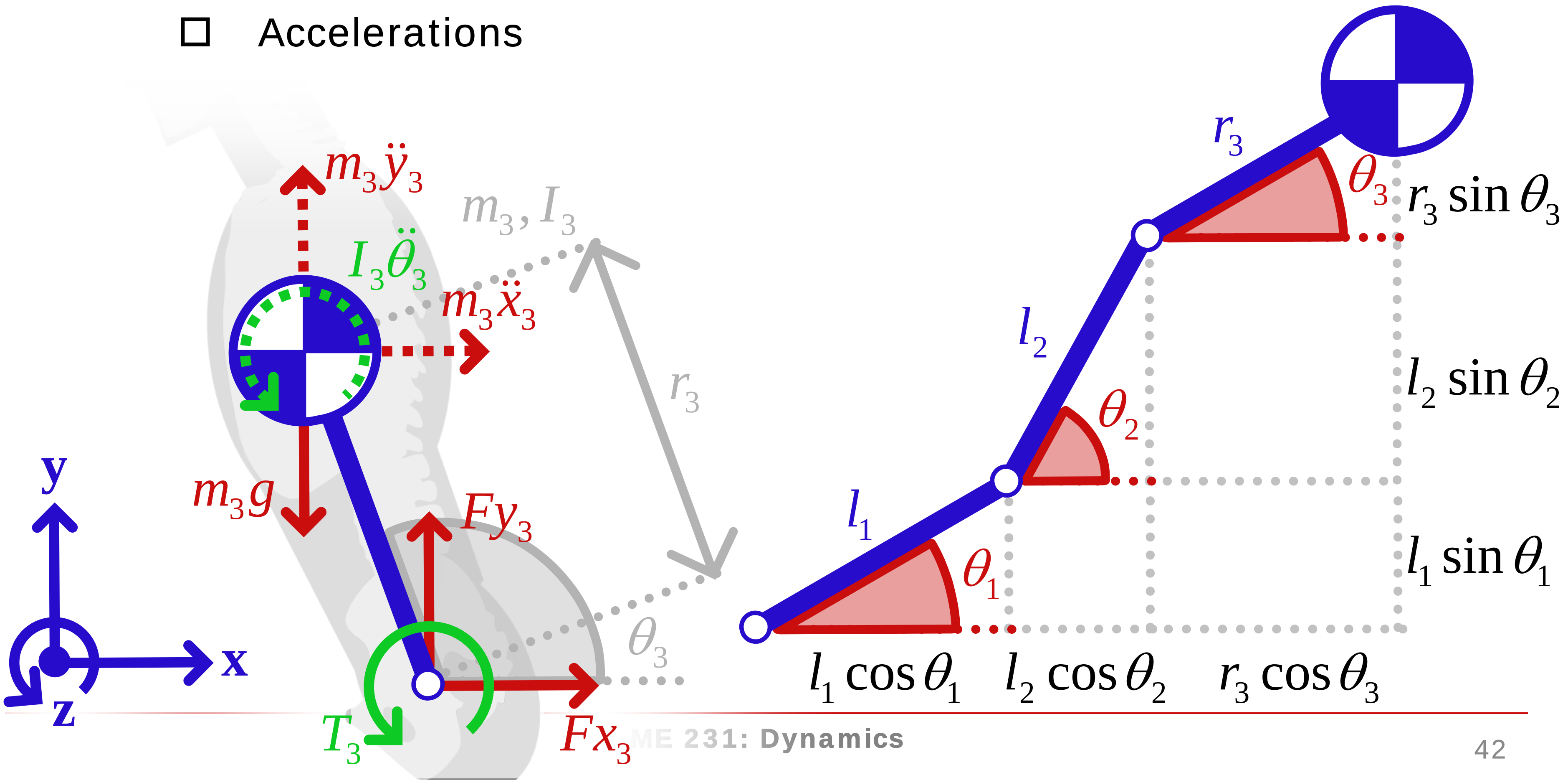
Segment 3 (head, arms, & trunk)

1. Create free body diagram
   - ☐ Reaction forces & moments
   - ☐ Distance forces & moments
   - ☐ Inertia forces & moments



Segment 3 (head, arms, & trunk)

2. Form motion quantities (kinematics)
   - ☐ Positions
   - ☐ Velocities
   - ☐ Accelerations



Segment 3 (head, arms, & trunk)

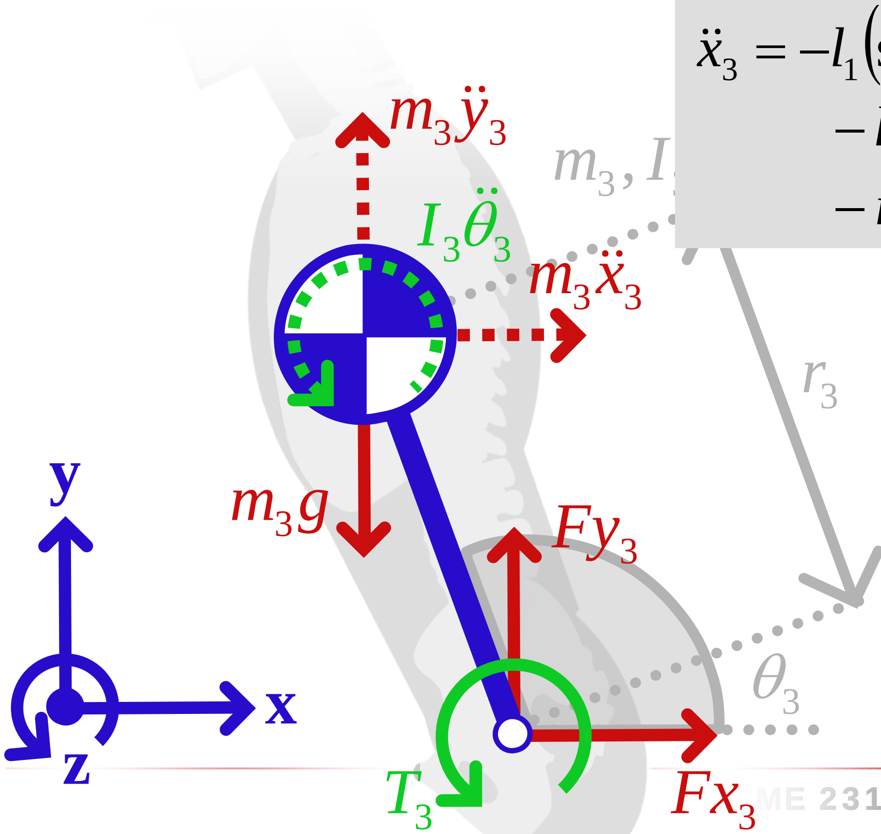
2. Form motion quantities (kinematics)
   - Positions
   - Velocities
   - Accelerations

$$x_3 = l_1 \mathrm{c}\theta_1 + l_2 \mathrm{c}\theta_2 + r_3 \mathrm{c}\theta_3 \qquad y_3 = l_1 \mathrm{s}\theta_1 + l_2 \mathrm{s}\theta_2 + r_3 \mathrm{s}\theta_3$$

$$\dot{x}_3 = -l_1 \mathrm{s}\theta_1 \dot{\theta}_1 - l_2 \mathrm{s}\theta_2 \dot{\theta}_2 - r_3 \mathrm{s}\theta_3 \dot{\theta}_3 \qquad \dot{y}_3 = l_1 \mathrm{c}\theta_1 \dot{\theta}_1 + l_2 \mathrm{c}\theta_2 \dot{\theta}_2 + r_3 \mathrm{c}\theta_3 \dot{\theta}_3$$

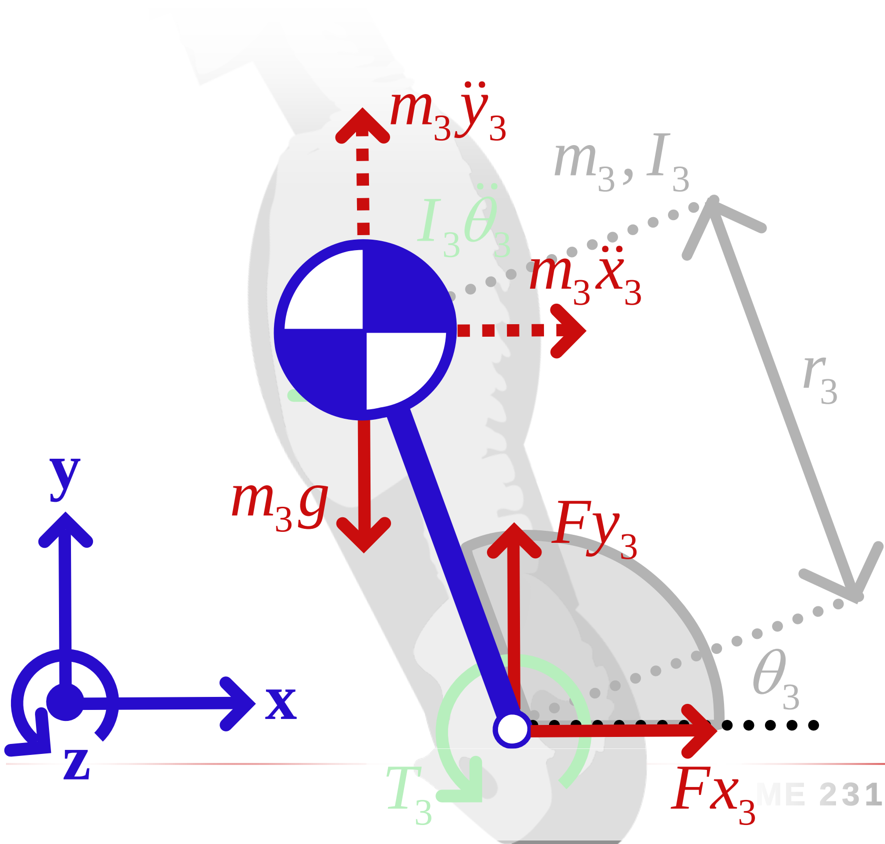
$$\ddot{x}_3 = -l_1\left(\mathrm{s}\theta_1 \ddot{\theta}_1 + \mathrm{c}\theta_1 \dot{\theta}_1^2\right) - l_2\left(\mathrm{s}\theta_2 \ddot{\theta}_2 + \mathrm{c}\theta_2 \dot{\theta}_2^2\right) - r_3\left(\mathrm{s}\theta_3 \ddot{\theta}_3 + \mathrm{c}\theta_3 \dot{\theta}_3^2\right)$$

$$\ddot{y}_3 = l_1\left(\mathrm{c}\theta_1 \ddot{\theta}_1 - \mathrm{s}\theta_1 \dot{\theta}_1^2\right) + l_2\left(\mathrm{c}\theta_2 \ddot{\theta}_2 - \mathrm{s}\theta_2 \dot{\theta}_2^2\right) + r_3\left(\mathrm{c}\theta_3 \ddot{\theta}_3 - \mathrm{s}\theta_3 \dot{\theta}_3^2\right)$$



Segment 3 (head, arms, & trunk)

3. Apply Newton's 2nd Law (kinetics)
   - ☐ Forces



Segment 3 (head, arms, & trunk)

3. Apply Newton's 2nd Law (kinetics)
   - Forces

$$\Sigma Fx = m_3\ddot{x}_3$$

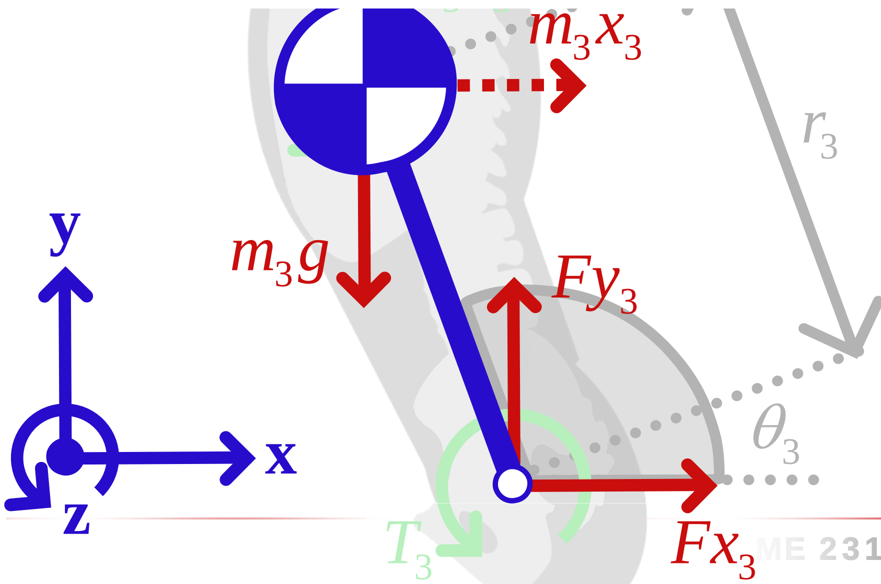
$$Fx_3 = m_3\ddot{x}_3$$

$$Fx_3 = -m_3\begin{pmatrix} l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) \\ +l_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right) \\ +r_3\left(\mathrm{s}\theta_3\ddot{\theta}_3 + \mathrm{c}\theta_3\dot{\theta}_3^2\right) \end{pmatrix} \tag{7}$$

$$\Sigma Fy = m_3\ddot{y}_3$$

$$Fy_3 - m_3 g = m_3\ddot{y}_3$$

$$Fy_3 - m_3 g = m_3\begin{pmatrix} l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) \\ +l_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) \\ +r_3\left(\mathrm{c}\theta_3\ddot{\theta}_3 - \mathrm{s}\theta_3\dot{\theta}_3^2\right) \end{pmatrix} \tag{8}$$



Segment 3 (head, arms, & trunk)

3. Apply Newton's 2nd Law (kinetics)
   - ☐ Forces

$$\Sigma Fx = m_3 \ddot{x}_3$$

$$Fx_3 = m_3 \ddot{x}_3$$

7

$$\boxed{Fx_3 = -m_3 \begin{pmatrix} l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) \\ +l_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right) \\ +r_3\left(\mathrm{s}\theta_3\ddot{\theta}_3 + \mathrm{c}\theta_3\dot{\theta}_3^2\right) \end{pmatrix}}$$

$$\Sigma Fy = m_3 \ddot{y}_3$$

$$Fy_3 - m_3 g = m_3 \ddot{y}_3$$

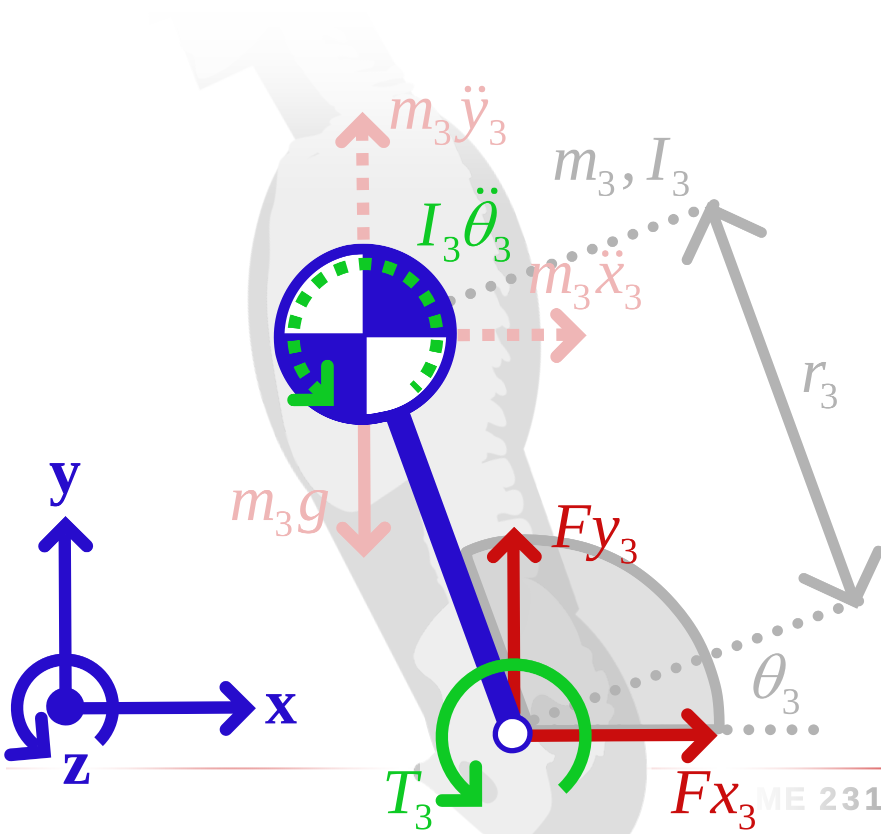
8

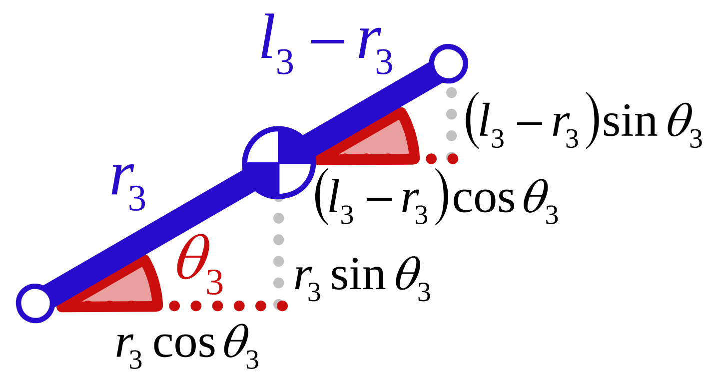
$$\boxed{Fy_3 - m_3 g = m_3 \begin{pmatrix} l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) \\ +l_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) \\ +r_3\left(\mathrm{c}\theta_3\ddot{\theta}_3 - \mathrm{s}\theta_3\dot{\theta}_3^2\right) \end{pmatrix}}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & r_1\mathrm{s}\theta_1 & -r_1\mathrm{c}\theta_1 & (l_1-r_1)\mathrm{s}\theta_1 & (l_1-r_1)\mathrm{c}\theta_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & r_2\mathrm{s}\theta_2 & 0 & -r_2\mathrm{c}\theta_2 & 0 & 1 & (l_2-r_2)\mathrm{s}\theta_2 & -(l_2-r_2)\mathrm{c}\theta_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Fx_1 \\ Fx_2 \\ Fy_1 \\ Fy_2 \\ T_1 \\ T_2 \\ Fx_3 \\ Fy_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} -m_1 r_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) \\ m_1\left(r_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + g\right) \\ I_1\ddot{\theta}_1 \\ -m_2\left(l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right)\right) \\ m_2\left(l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) + g\right) \\ I_2\ddot{\theta}_2 \\ -m_3\left(l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) + l_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right) + r_3\left(\mathrm{s}\theta_3\ddot{\theta}_3 + \mathrm{c}\theta_3\dot{\theta}_3^2\right)\right) \\ m_3\left(l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + l_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) + r_3\left(\mathrm{c}\theta_3\ddot{\theta}_3 - \mathrm{s}\theta_3\dot{\theta}_3^2\right) + g\right) \end{bmatrix}$$

Segment 3 (head, arms, & trunk)

3. Apply Newton's 2nd Law (kinetics)
   - Moments



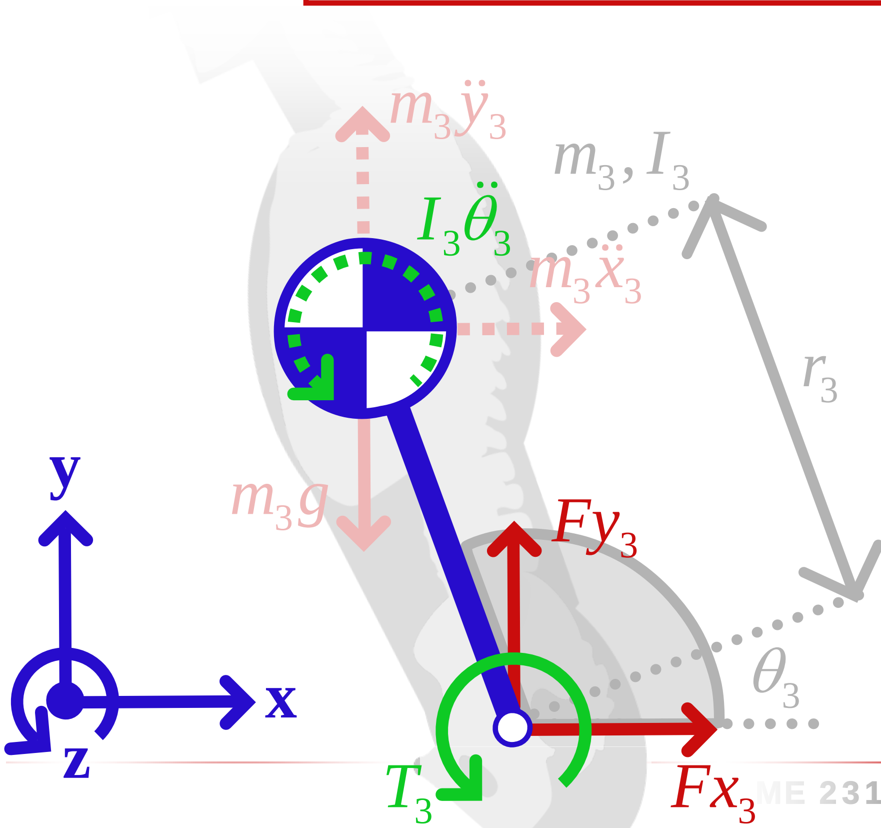


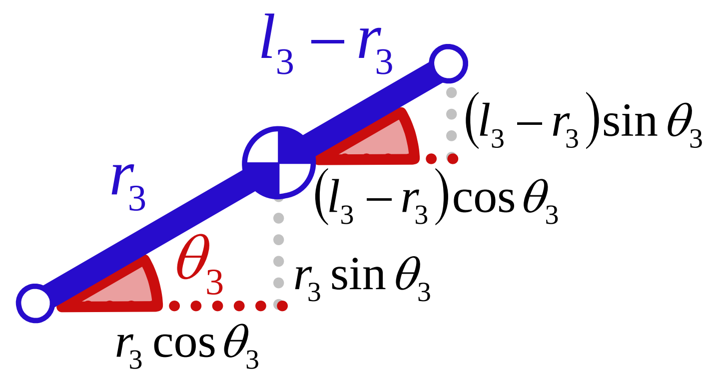
Segment 3 (head, arms, & trunk)

3. Apply Newton's 2nd Law (kinetics)
   - ☐ Moments

$$\Sigma M = I_3\ddot{\theta}_3$$

$$\boxed{T_3 + Fx_3 r_3 \mathrm{s}\theta_3 - Fy_3 r_3 \mathrm{c}\theta_3 = I_3\ddot{\theta}_3}$$ 9





Segment 3 (head, arms, & trunk)

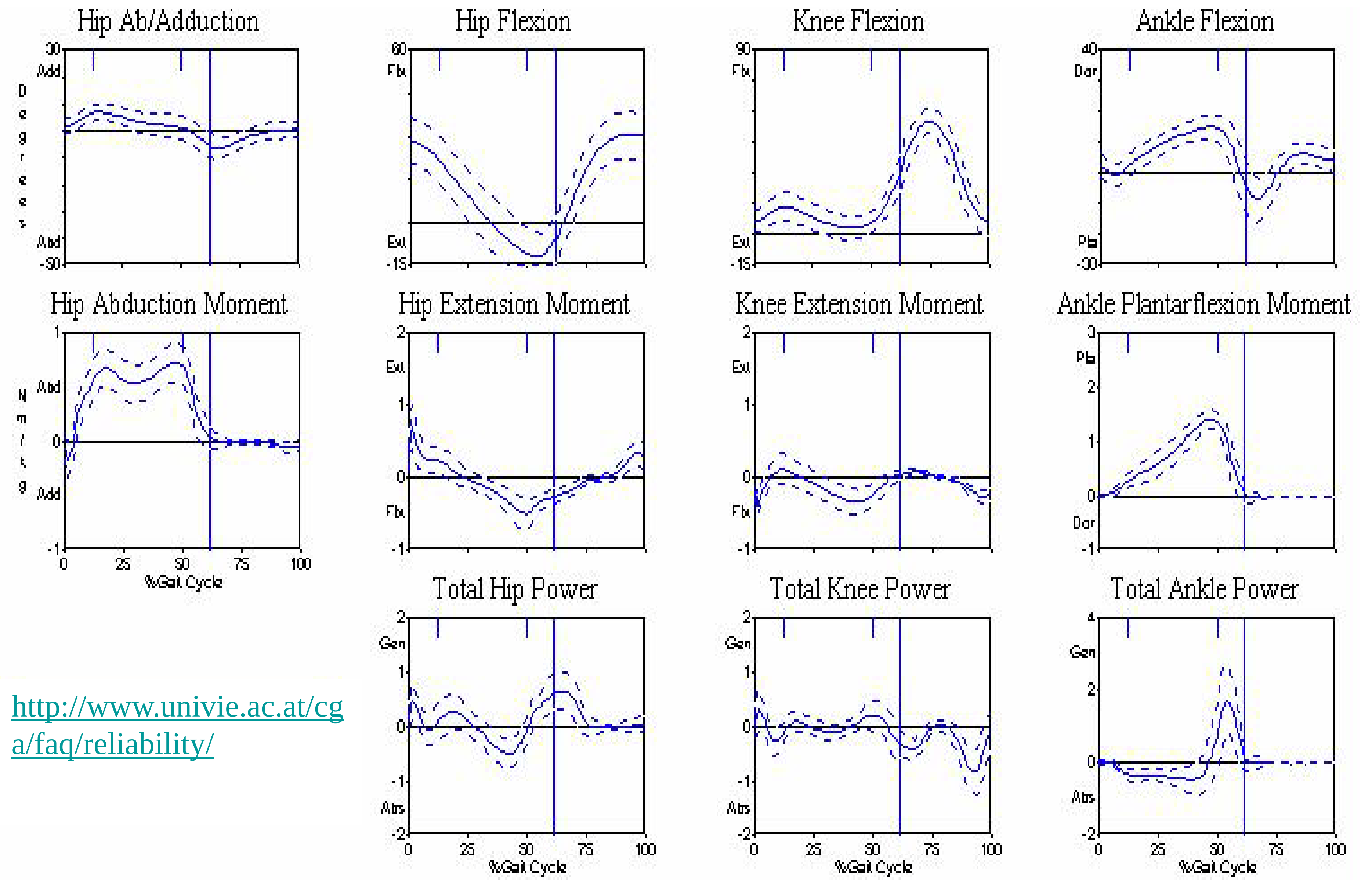
3. Apply Newton's 2nd Law (kinetics)
   - Moments

$$\Sigma M = I_3\ddot{\theta}_3$$

$$\boxed{T_3 + Fx_3 r_3 \mathrm{s}\theta_3 - Fy_3 r_3 \mathrm{c}\theta_3 = I_3\ddot{\theta}_3} \quad 9$$

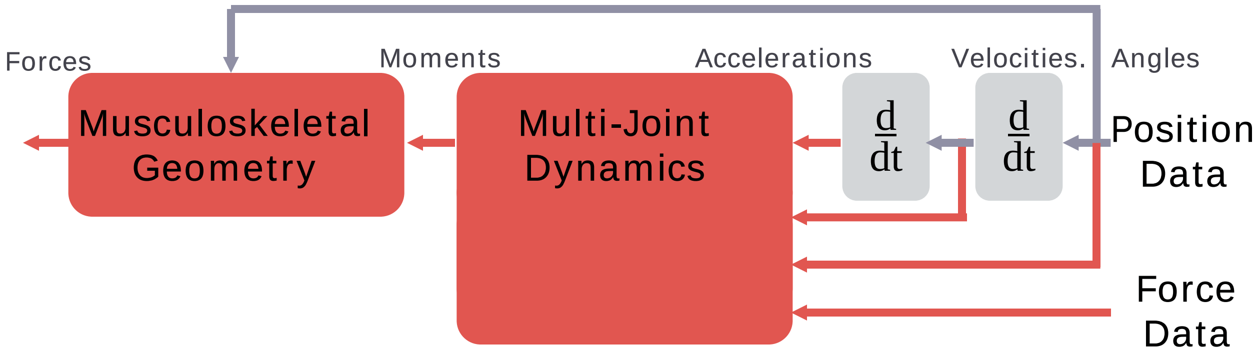
$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array}
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & r_1\mathrm{s}\theta_1 & -r_1\mathrm{c}\theta_1 & (l_1-r_1)\mathrm{s}\theta_1 & (l_1-r_1)\mathrm{c}\theta_1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\
0 & r_2\mathrm{s}\theta_2 & 0 & -r_2\mathrm{c}\theta_2 & 0 & 1 & (l_2-r_2)\mathrm{s}\theta_2 & -(l_2-r_2)\mathrm{c}\theta_2 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & r_3\mathrm{s}\theta_3 & -r_3\mathrm{c}\theta_3 & 1
\end{bmatrix}
\begin{bmatrix} Fx_1\\ Fx_2\\ Fy_1\\ Fy_2\\ T_1\\ T_2\\ Fx_3\\ Fy_3\\ T_3 \end{bmatrix}
=
\begin{bmatrix}
-m_1 r_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right)\\
m_1\left(r_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + g\right)\\
I_1\ddot{\theta}_1\\
-m_2\left(l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right)\right)\\
m_2\left(l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + r_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) + g\right)\\
I_2\ddot{\theta}_2\\
-m_3\left(l_1\left(\mathrm{s}\theta_1\ddot{\theta}_1 + \mathrm{c}\theta_1\dot{\theta}_1^2\right) + l_2\left(\mathrm{s}\theta_2\ddot{\theta}_2 + \mathrm{c}\theta_2\dot{\theta}_2^2\right) + r_3\left(\mathrm{s}\theta_3\ddot{\theta}_3 + \mathrm{c}\theta_3\dot{\theta}_3^2\right)\right)\\
m_3\left(l_1\left(\mathrm{c}\theta_1\ddot{\theta}_1 - \mathrm{s}\theta_1\dot{\theta}_1^2\right) + l_2\left(\mathrm{c}\theta_2\ddot{\theta}_2 - \mathrm{s}\theta_2\dot{\theta}_2^2\right) + r_3\left(\mathrm{c}\theta_3\ddot{\theta}_3 - \mathrm{s}\theta_3\dot{\theta}_3^2\right) + g\right)\\
I_3\ddot{\theta}_3
\end{bmatrix}$$

Normal Gait: Joint Kinematics & Kinetics



http://www.univie.ac.at/cga/faq/reliability/

The Inverse Dynamics Problem

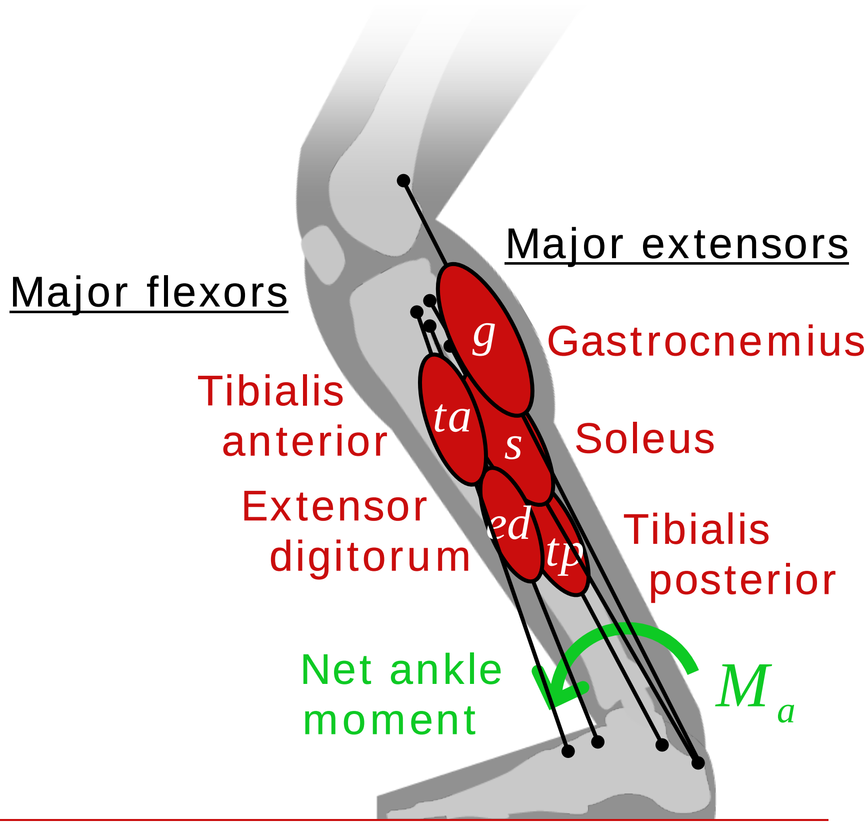


- ✓ Derive equations of motion from model of system
- ✓ Solve equations of motion with and without external forces
- • Use musculoskeletal geometry and assumptions about force distribution to estimate individual muscle forces

Net Joint Moments from Muscle Forces

Net joint moments are produced by multiple muscles (previously assumed to be ideal torque actuators)

What factors will affect how much a muscle contributes to the net moment?



The "Distribution" Problem

$$M_j = \sum \text{muscle moments} + \sum \text{moments due to other structures}$$

number of flexors

number of extensors

$$M_j = \underbrace{\sum_{f=1}^{nf} F_f r_f}_{\text{flexion moment}} - \underbrace{\sum_{e=1}^{ne} F_e r_e}_{\text{extension moment}}$$

moment arm

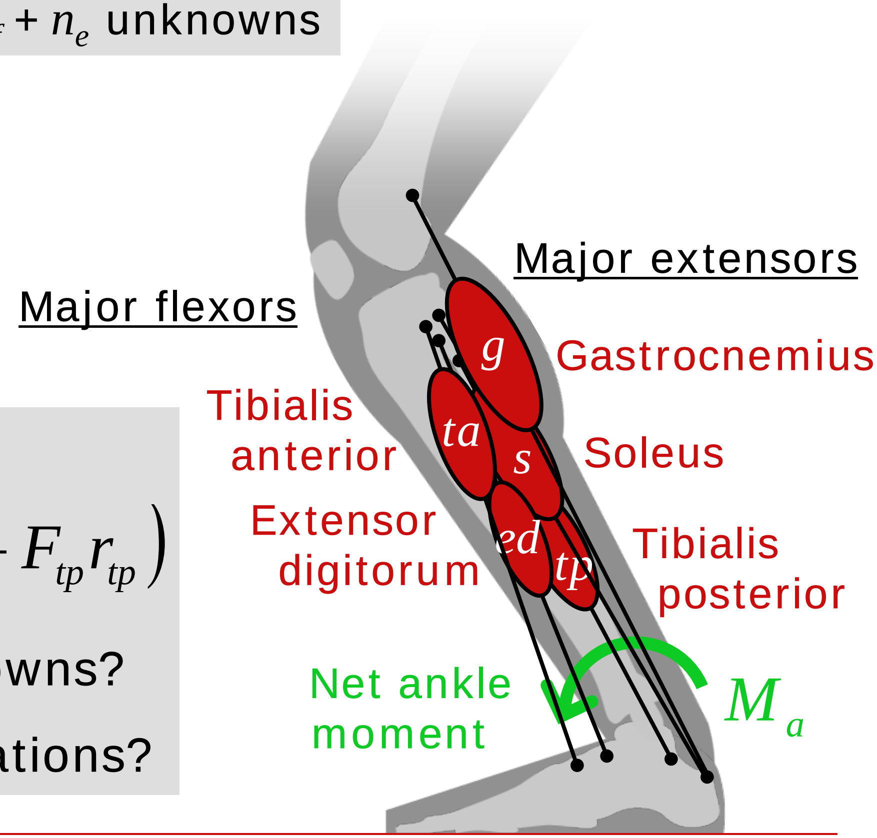
1 equation with $n_f + n_e$ unknowns

Ankle example

$$M_a = \left(F_{ta} r_{ta} + F_{ed} r_{ed}\right) - \left(F_g r_g + F_s r_s + F_{tp} r_{tp}\right)$$

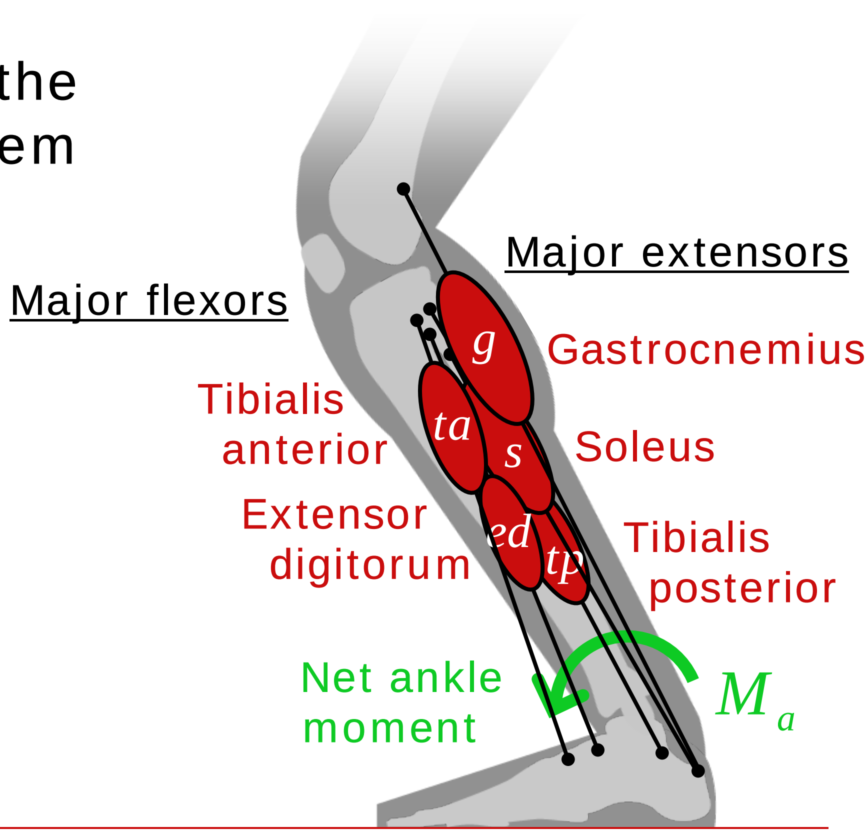
Can we reduce the number of unknowns?

Can we increase the number of equations?



Static Optimization

- Select a criterion to minimize or maximize subject to a set of constraints
- Criterion and constraints are a function of muscle forces
- Criterion attempts to capture the goal of the neural control system
  - Minimize muscle force?
  - Minimize muscle stress?



Static Optimization Example

minimize $f(F_m)$ Function of muscle forces

subject to $M_a(t) = \left[F_{ta}(t)r_{ta}(t) + F_{ed}(t)r_{ed}(t)\right] - \left[F_g(t)r_g(t) + F_s(t)r_s(t) + F_{tp}(t)r_{tp}(t)\right]$

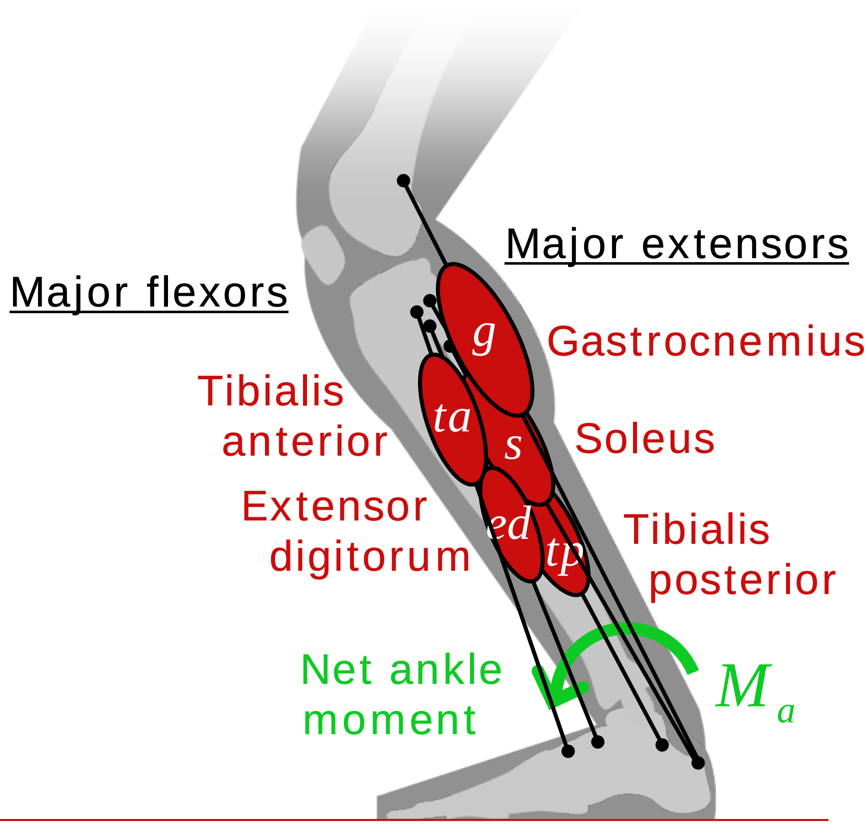
$F_{ta}(t) \leq 900\text{N}$

$F_{ed}(t) \leq 800\text{N}$

$F_g(t) \leq 1500\text{N}$

$F_s(t) \leq 2500\text{N}$

$F_{tp}(t) \leq 1500\text{N}$



Static Optimization Example

minimize $\sum_{m=1}^{nm} F_m$ Total muscle force

subject to $M_a(t) = \left[F_{ta}(t) r_{ta}(t) + F_{ed}(t) r_{ed}(t)\right] - \left[F_g(t) r_g(t) + F_s(t) r_s(t) + F_{tp}(t) r_{tp}(t)\right]$

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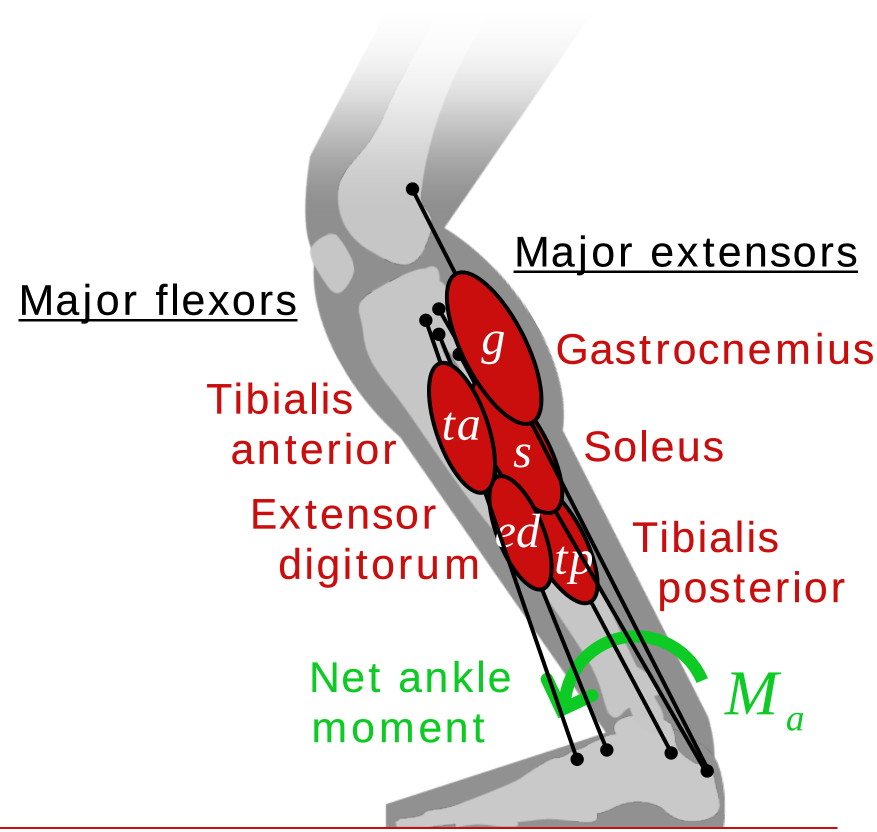

Static Optimization Example

minimize $\displaystyle\sum_{m=1}^{nm} \frac{F_m}{PCSA_m}$ Total muscle stress

subject to $M_a(t) = \left[F_{ta}(t)r_{ta}(t) + F_{ed}(t)r_{ed}(t)\right] - \left[F_g(t)r_g(t) + F_s(t)r_s(t) + F_{tp}(t)r_{tp}(t)\right]$

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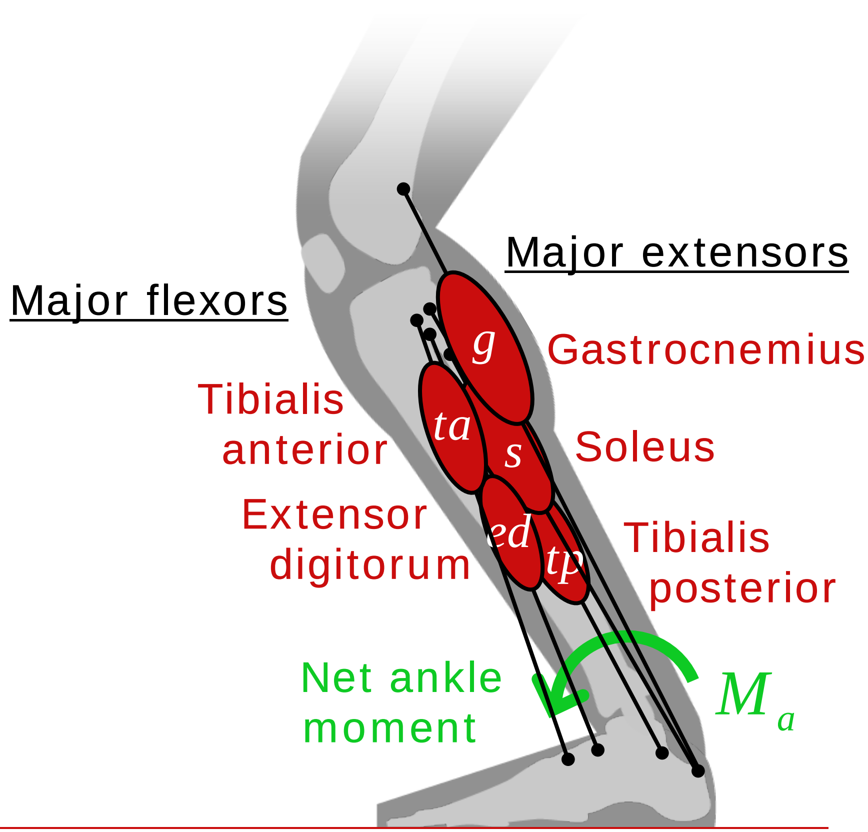

Static Optimization Example

minimize $\sum_{m=1}^{nm}\left(\frac{F_m}{PCSA_m}\right)^3$ Total (muscle stress)³ ~ metabolic energy

subject to $M_a(t) = \left[F_{ta}(t)r_{ta}(t) + F_{ed}(t)r_{ed}(t)\right] - \left[F_g(t)r_g(t) + F_s(t)r_s(t) + F_{tp}(t)r_{tp}(t)\right]$

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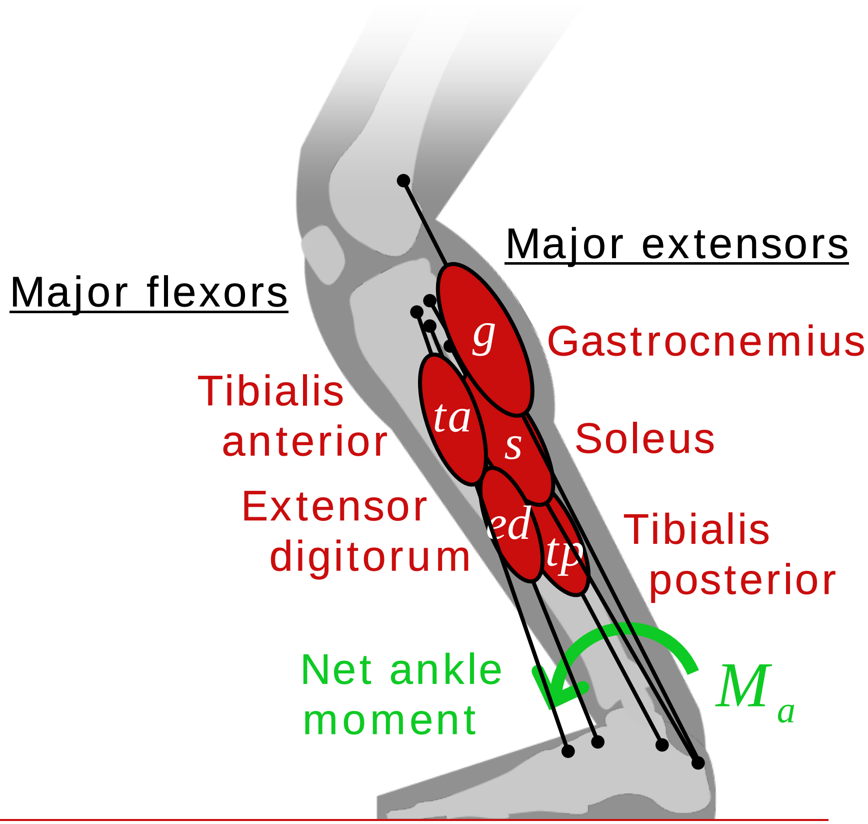


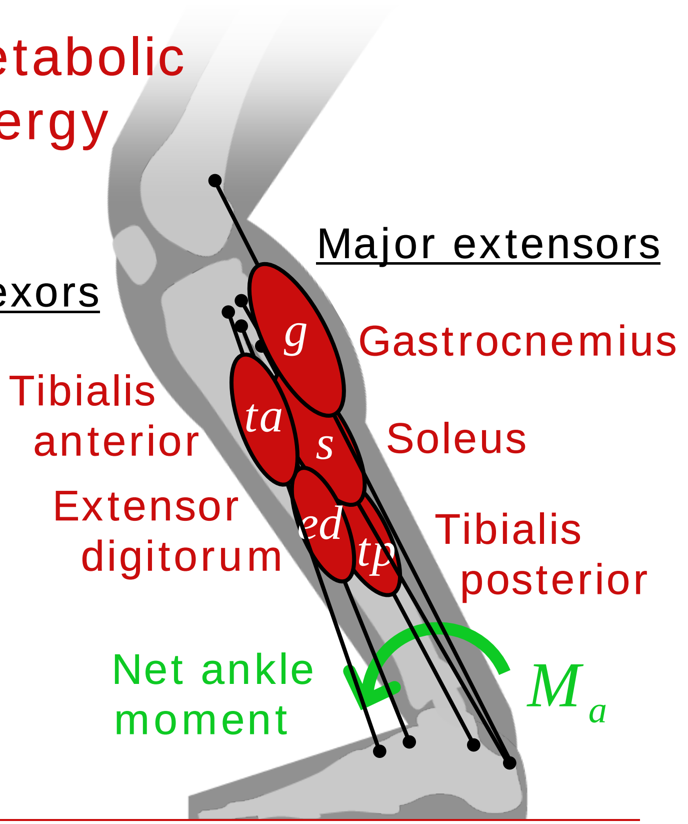
Consider These Functions

$$\sum_{m=1}^{nm} F_m$$ Total muscle force

$$\sum_{m=1}^{nm} \frac{F_m}{PCSA_m}$$ Total muscle stress

$$\sum_{m=1}^{nm} \left(\frac{F_m}{PCSA_m} \right)^3$$ Total (muscle stress)³ ~ metabolic energy

Difficult to define and validate a good criterion



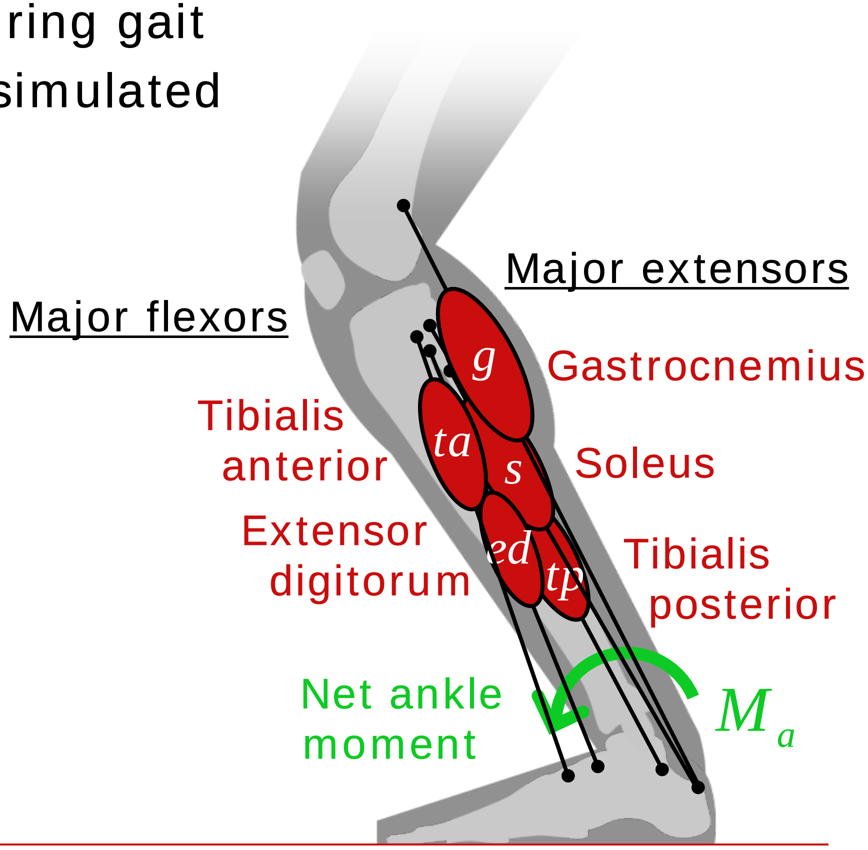
Possible validations

- Use output to drive a forward dynamic simulation
- Compare qualitatively to experimental EMG
- Compare to measured forces (instrumented hip implant, buckle transducer in tendon)

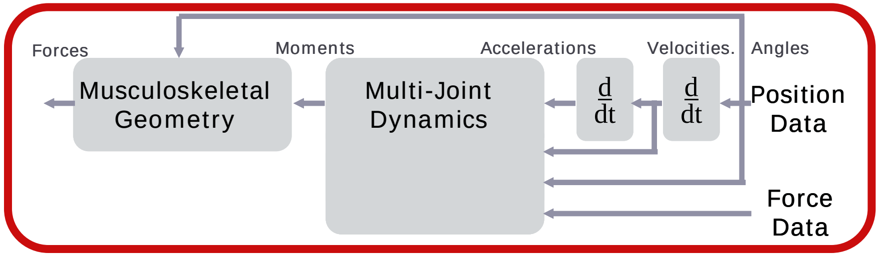
Dynamic Optimization

- Instead of optimizing over an instant can optimize a criterion for a motion
  - maximize height reached during a jump
  - minimize metabolic energy during gait
  - minimize difference between simulated and desired motion

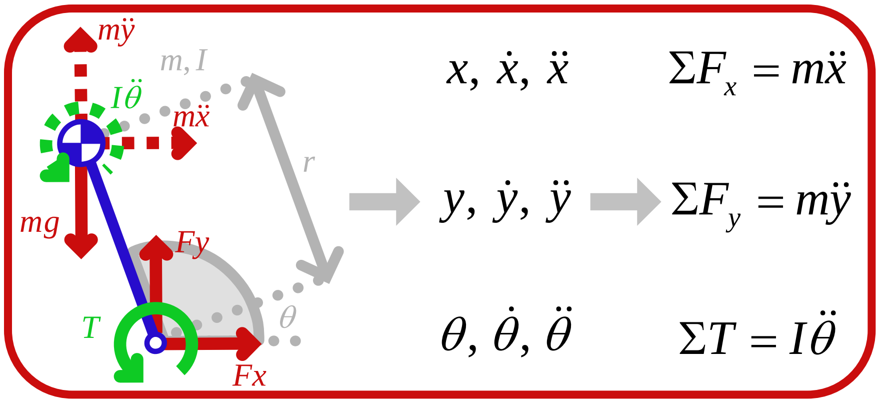
- Can take dynamic properties into account
  - force-length-velocity properties of muscle
  - activation dynamics of muscle



Main Points of the Day

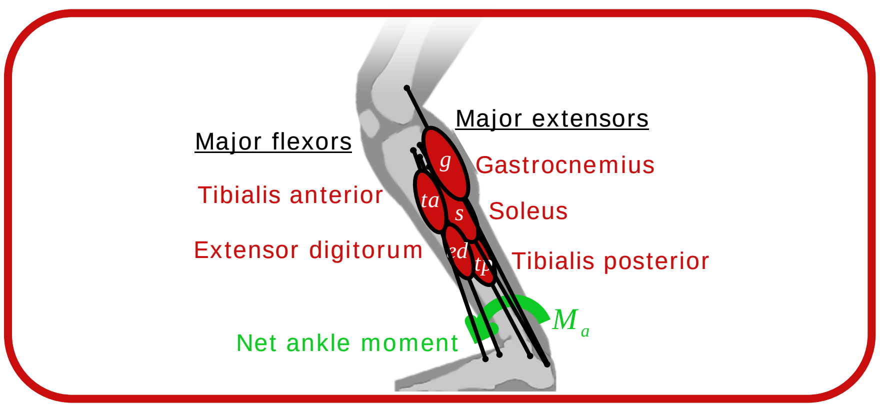


Inverse dynamics problem is useful to solve



3 steps are required for inverse dynamics

- Create free body diagram
- Solve for kinematics
- Solve for kinetics



Muscle force distribution is a problem

- More muscles than DOFs
- Optimization is useful to find a solution

For Next Time…

- Continue Homework #9 due on ***Wednesday (10/26)***
- Read Chapter 3, Articles 3/8 & 3/9